Uniformly Lipschitzian mappings in modular function spaces

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0. Introduction

The theory of modular spaces was initiated by Nakano [14] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [13] in 1959. Defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied (see, for instance, [15]). Another direction is based on considering an abstractly given functional which controls the growth of the functions. Even though a metric is not defined, many problems in fixed point theory for nonexpansive mappings can be reformulated in modular spaces (see, for instance, [8] and references therein). In this paper, we study the existence of fixed points for a more general class of mappings: uniformly Lipschitzian mappings. Fixed point theorems for this class of mappings in Banach spaces have been studied in [2,3] and in metric spaces in [11,12] (for further information about this subject, see [1, Chapter VIII] and references therein). The main tool in our approach is the coefficient of normal structure $\tilde{N}(L_p)$. We prove that under suitable conditions a $k$-uniformly Lipschitzian mapping has a fixed point if $k < (\tilde{N}(L_p))^{-1/2}$. In the last section we show a class of modular spaces where $\tilde{N}(L_p) < 1$ and so, the above theorem can be successfully applied.

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1. Preliminaries

We start by recording a brief collection of basic concepts and facts of modular spaces as formulated by Kozlowski. For more details the reader is referred to [7,8,10,13].

Let \( \Omega \) be a nonempty set and \( \Sigma \) be a nontrivial \( \sigma \)-algebra of subsets of \( \Omega \). Let \( \mathcal{P} \) be a \( \delta \)-ring of subsets of \( \Sigma \), such that \( E \cap A \in \mathcal{P} \) for any \( E \in \mathcal{P} \) and \( A \in \Sigma \). Let us assume that there exists an increasing sequence of sets \( K_n \in \mathcal{P} \) such that \( \Omega = \bigcup K_n \). In other words, the family \( \mathcal{P} \) plays the role of the \( \delta \)-ring of subsets of finite measure. By \( \mathcal{E} \) we denote the linear space of all simple functions with supports from \( \mathcal{P} \). By \( \mathcal{M} \) we will denote the space of all measurable functions, i.e. all functions \( f: \Omega \to \mathbb{R} \) such that there exists a sequence \( \{g_n\} \in \mathcal{E} \), \( |g_n| \leq |f| \) and \( g_n(\omega) \to f(\omega) \) for all \( \omega \in \Omega \). By \( 1_A \) we denote the characteristic function of the set \( A \).

**Definition 1.1.** A functional \( \rho: \mathcal{E} \times \Sigma \to [0, \infty] \) is called a function modular if

\[
\begin{align*}
(P_1) \quad & \rho(0, E) = 0 \text{ for any } E \in \Sigma, \\
(P_2) \quad & \rho(f, E) \leq \rho(g, E) \text{ whenever } |f(\omega)| \leq |g(\omega)| \text{ for any } \omega \in \Omega, \ f, g \in \mathcal{E} \text{ and } E \in \Sigma, \\
(P_3) \quad & \rho(f, \cdot): \Sigma \to [0, \infty] \text{ is a } \sigma \text{-subadditive measure for every } f \in \mathcal{E}, \\
(P_4) \quad & \rho(x, A) \to 0 \text{ as } x \text{ decreases to } 0 \text{ for every } A \in \mathcal{P}, \text{ where } \rho(x, A) = \rho(x1_A, A), \\
(P_5) \quad & \text{if there exists } x > 0 \text{ such that } \rho(x, A) = 0, \text{ then } \rho(\beta, A) = 0 \text{ for every } \beta > 0, \\
(P_6) \quad & \text{for any } x > 0 \rho(x, \cdot) \text{ is order continuous on } \mathcal{P}, \text{ that is } \rho(x, A_n) \to 0 \text{ if } \{A_n\} \in \mathcal{P} \text{ and decreases to } \emptyset.
\end{align*}
\]

The definition of \( \rho \) is then extended to \( f \in \mathcal{M} \) by

\[
\rho(f, E) = \sup\{\rho(g, E); g \in \mathcal{E}, \ |g(\omega)| \leq |f(\omega)| \ \omega \in \Omega\}.
\]

A set \( E \) is said to be \( \rho \)-null if \( \rho(x, E) = 0 \) for every \( x > 0 \). For the sake of simplicity we write \( \rho(f) \) instead of \( \rho(f, \Omega) \).

It is easy to see that the functional \( \rho: \mathcal{M} \to [0, \infty] \) is a modular because it satisfies the following properties:

1. \( \rho(f) = 0 \) iff \( f = 0 \) \( \rho \)-a.e.
2. \( \rho(\alpha f) = \rho(f) \) for every scalar \( \alpha \) with \( |\alpha| = 1 \) and \( f \in \mathcal{M} \).
3. \( \rho(\alpha f + \beta g) \leq \rho(f) + \rho(g) \) if \( \alpha + \beta = 1, \ \alpha \geq 0, \beta \geq 0 \) and \( f, g \in \mathcal{M} \).

In addition, if the following property is satisfied

\[
\rho(\alpha f + \beta g) \leq \alpha \rho(f) + \beta \rho(g) \text{ if } \alpha + \beta = 1; \ \alpha \geq 0, \beta \geq 0 \text{ and } f, g \in \mathcal{M},
\]

we say that \( \rho \) is a convex modular.

The modular \( \rho \) defines a corresponding modular space, i.e the vector space \( L_\rho \) given by

\[
L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.
\]

We can also consider the space \( E_\rho = \{f \in \mathcal{M}; \rho(\alpha f, A_n) \to 0 \text{ as } n \to \infty \text{ for every } A_n \in \Sigma \text{ that decreases to } \emptyset \text{ and } \alpha > 0\} \).

A function modular is said to satisfy the \( \Delta_2 \)-condition if \( \sup_{n \geq 1} \rho(2f_n, D_k) \to 0 \) as \( k \to \infty \) whenever \( \{f_n\}_{n \geq 1} \subset \mathcal{M}, D_k \in \Sigma \text{ decreases to } \emptyset \text{ and } \sup_{n \geq 1} \rho(f_n, D_k) \to 0 \) as \( k \to \infty \). We know (see [10]) that \( E_\rho = L_\rho \) when \( \rho \) satisfies the \( \Delta_2 \)-condition. When \( \rho \)
is convex, the formula
\[ \| f \|_\rho = \inf \left\{ \alpha > 0; \rho \left( \frac{f}{\alpha} \right) \leq 1 \right\} \]
defines a norm in the modular space \( L_\rho \) which is frequently called the Luxemburg norm.

**Definition 1.2.** (1) The sequence \( \{ f_n \} \subset L_\rho \) is said to be \( \rho \)-convergent to \( f \in L_\rho \) if \( \rho(f_n - f) \to 0 \) as \( n \to \infty \).

(2) The sequence \( \{ f_n \} \subset L_\rho \) is said to be \( \rho \)-a.e. convergent to \( f \in L_\rho \) if the set \( \{ \omega \in \Omega; f_n(\omega) \rightharpoonup f(\omega) \} \) is \( \rho \)-null.

(3) The sequence \( \{ f_n \} \subset L_\rho \) is said to be \( \rho \)-Cauchy if \( \rho(f_n - f_m) \to 0 \) as \( n \) and \( m \) go to \( \infty \).

(4) A subset \( C \) of \( L_\rho \) is called \( \rho \)-closed if the \( \rho \)-limit of a \( \rho \)-convergent sequence of \( C \) always belongs to \( C \).

(5) A subset \( C \) of \( L_\rho \) is called \( \rho \)-a.e. sequentially closed if the \( \rho \)-a.e. limit of a \( \rho \)-a.e. convergent sequence of \( C \) always belongs to \( C \).

(6) A subset \( C \) of \( L_\rho \) is called \( \rho \)-a.e. sequentially compact if every sequence in \( C \) has a \( \rho \)-a.e. convergent subsequence in \( C \).

(7) A subset \( C \) of \( L_\rho \) is called \( \rho \)-bounded if
\[ \delta_\rho(C) = \sup \{ \rho(f - g); f, g \in C \} < \infty. \]

Let \( B \) be a bounded subset of \( L_\rho \). We define the \( \rho \)-ball of center \( f \in L_\rho \) and radius \( r > 0 \) by \( B(f, r) = \{ g \in L_\rho; \rho(g - f) \leq r \} \). We will denote \( r(f, B) = \sup \{ \rho(f - g); g \in B \} \), \( \delta(B) = \sup \{ r(f, B); f \in B \} \), \( R(B) = \inf \{ r(f, B); f \in B \} \). We define the admissible hull of \( B \) as the intersection of all \( \rho \)-ball containing \( B \), i.e.
\[ ad(B) = \bigcap \{ A; B \subset A \subset L_\rho, \text{ where } A \text{ is a } \rho \text{-ball} \}. \]

\( B \) is said admissible if \( ad(B) = B \). We define the normal structure coefficient \( \hat{N}(L_\rho) \) of \( L_\rho \) by
\[ \hat{N}(L_\rho) = \sup \{ R(B)/\delta(B); B \text{ is admissible, } \rho \text{-bounded and } \rho \text{-a.e. sequentially compact} \}. \]

The useful following proposition is easily seen:

**Proposition 1.1.** Let \( B \) be a \( \rho \)-bounded subset of \( L_\rho \) and \( f \in L_\rho \). Then
(1) \( r(f, ad(B)) = r(f, B) \).
(2) \( \delta(ad(B)) = \delta(B) \).

We say that \( \rho \) satisfies the \( \Delta_2 \)-type condition if there exists \( K > 0 \) such that \( \rho(2f) \leq K\rho(f) \) for all \( f \in L_\rho \). In general, \( \Delta_2 \)-type condition and \( \Delta_2 \)-condition are not equivalent, even though it is obvious that \( \Delta_2 \)-type condition implies \( \Delta_2 \)-condition. Assume that \( \rho \) is convex and satisfies the \( \Delta_2 \)-type condition. We define a growth function
The following properties of the growth function can be easily seen.

**Lemma 1.1.** Let \( \rho \) be a convex function modular satisfying the \( \Delta_2 \)-type condition. Then the growth function \( \omega \) has the following properties:

1. \( \omega(t) < \infty, \quad \forall t \in [0, \infty) \).
2. \( \omega : [0, \infty) \to [0, \infty) \) is a convex, strictly increasing function. So, it is continuous.
3. \( \omega(x\beta) \leq \omega(x)\omega(\beta); \quad \forall x, \beta \in [0, \infty) \).
4. \( \omega^{-1}(x)\omega^{-1}(\beta) \leq \omega^{-1}(x\beta); \quad \forall x, \beta \in [0, \infty) \), where \( \omega^{-1} \) is the function inverse of \( \omega \).

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

**Lemma 1.2** (Dominguez Benavides et al. [4]). Let \( \rho \) be a convex function modular satisfying the \( \Delta_2 \)-type condition. Then

\[
\|f\|_{\rho} \leq \frac{1}{\omega^{-1}(1/\rho(f))} \quad \text{whenever } f \in L_\rho.
\]

The following lemma can be found in [7].

**Lemma 1.3.** Let \( \rho \) be a function modular satisfying the \( \Delta_2 \)-condition and \( \{f_n\}_n \) be a sequence in \( L_\rho \) such that \( f_n^{\rho-a.e.} f \in L_\rho \) and there exists \( k > 1 \) such that \( \sup_n \rho(k(f_n - f)) < \infty \). Then,

\[
\liminf_{n \to \infty} \rho(f_n - g) = \liminf_{n \to \infty} \rho(f_n - f) + \rho(f - g) \quad \text{for all } g \in L_\rho.
\]

**Lemma 1.4.** Let \( \rho \) be a modular function satisfying the \( \Delta_2 \)-type condition. Let \( B \) be a \( \rho \)-a.e. sequentially closed and \( \rho \)-bounded subset of \( L_\rho \). Let \( \{g_n\}_n \) be a sequence in \( B \) such that \( g_n^{\rho-a.e.} g \). Then,

1. \( \rho(g) \leq \liminf_{n \to \infty} \rho(g_n) \).
2. \( B(0, r) \cap B \) is \( \rho \)-a.e. sequentially closed.
3. \( \text{ad}(A) \cap B \) is \( \rho \)-a.e. sequentially closed, for all \( A \subset L_\rho \).

**Proof.** Condition (1) is a straightforward consequence of Lemma 1.3 applied to the sequence \( g_n^{\rho-a.e.} g \) and the null function. Conditions (2) and (3) can be easily deduced from (1). \( \square \)

2. **Fixed point for uniformly Lipschitzian mappings**

The following lemma is the key of our fixed point result.
Lemma 2.1. Let $\rho$ be a modular function satisfying the $\Delta_2$-type condition and $B$ a $\rho$-bounded and $\rho$-a.e. sequentially compact subset of $L_\rho$. Let $\{f_n\}_n$ and $\{g_n\}_n$ be sequences in $B$. Then, there exists $g \in \bigcap_{n=1}^\infty \text{ad}(g_j, j \geq n) \cap B$ such that

$$\limsup_{n \to \infty} \rho(g - f_n) \leq \limsup_{j \to \infty} \limsup_{n \to \infty} \rho(g_j - f_n).$$

Proof. Let $\{f_n\}_n$ and $\{g_n\}_n$ be sequences in $B$. We define $\theta(h) = \limsup_{n \to \infty} \rho(h - f_n)$ for all $h \in B$. Since $B$ is $\rho$-sequentially compact and $\rho$-bounded, there exist a subsequence $\{g_{\phi(n)}\}_n \subset \{g_n\}_n$ such that $g_{\phi(n)} \rightarrow^{\rho\text{-a.e.}} g$ and a subsequence $\{f_{\psi(n)}\}_n \subset \{f_n\}_n$ such that $\lim_{n \to \infty} \rho(f_{\psi(n)} - g) = \limsup_{n \to \infty} \rho(f_n - g)$ and $f_{\psi(n)} \rightarrow^{\rho\text{-a.e.}} f \in B$. Since $g_{\phi(n)} \in \text{ad}(g_j, j \geq n) \cap B$ which is $\rho$-a.e. sequentially closed (by property (3) of Lemma 1.4) and $g_{\phi(n)} \rightarrow^{\rho\text{-a.e.}} g$, we obtain $g \in \text{ad}(g_j, j \geq n) \cap B$ for all $n \geq 1$. We will see that $\theta(g) \leq \limsup_{j \to \infty} \theta(g_j)$. Indeed, from Lemma 2.3 we have $\theta(g_j) = \limsup_{n \to \infty} \rho(f_n - g_j) \geq \liminf_{n \to \infty} \rho(f_{\psi(n)} - g_j) = \liminf_{n \to \infty} \rho(f_{\psi(n)} - f) + \rho(f - g_j)$. Thus, again using Lemma 2.3, we obtain

$$\limsup_{j \to \infty} \theta(g_j) \geq \liminf_{n \to \infty} \rho(f_{\psi(n)} - f) + \limsup_{j \to \infty} \rho(f - g_j) \geq \liminf_{n \to \infty} \rho(f_{\psi(n)} - f) + \liminf_{j \to \infty} \rho(g_{\phi(j)} - g) \geq \liminf_{n \to \infty} \rho(f_{\psi(n)} - f) + \liminf_{j \to \infty} \rho(g_{\phi(j)} - g) + \rho(f - g).$$

On the other hand, $\theta(g) = \limsup_{n \to \infty} \rho(f_n - g) = \liminf_{n \to \infty} \rho(f_{\psi(n)} - g) = \liminf_{n \to \infty} \rho(f_{\psi(n)} - f) + \rho(f - g)$. Therefore, $\theta(g) \leq \limsup_{j \to \infty} \theta(g_j)$. \hfill \Box

The following lemma is inspired on [2] where a similar lemma is proved in reflexive Banach spaces (see also [12, Lemma 6] for a version in metric spaces with additional properties).

Lemma 2.2. Let $\rho$ be a function modular satisfying the $\Delta_2$-type condition and $B$ an admissible, $\rho$-a.e. sequentially compact and $\rho$-bounded subset of $L_\rho$. Let $\{f_n\}$ be a sequence in $B$ and $c$ a constant such that $c > \tilde{N}(L_\rho)$. Then there exists $f \in B$ such that

1. $\limsup_{n \to \infty} \rho(f - f_n) \leq c\delta(\{f_n\}_n)$.
2. $\rho(f - g) \leq \limsup_{n \to \infty} \rho(f_n - g)$ for all $g \in B$.

Proof. Let $\{f_n\}$ be a sequence of $B$. Denote $A_m = \text{ad}(f_j; j \geq m) \subset B$ and $A = \bigcap_{m=1}^\infty A_m$. Since $B$ is $\rho$-a.e. sequentially compact, there exists a subsequence of $\{f_n\}_n$ $\rho$-a.e. convergent, say to $h$. It is clear that $h \in A$ and so $A \neq \emptyset$. Furthermore, from Proposition 1.1(2), we have $\delta(A_m) \leq \delta(\{f_n\}_n)$. On the other hand, for any $f \in A$ and $g \in B$ we have $\rho(g - f) \leq r(g, A) \leq r(g, A_m) = r(g, \{f_j; j \geq n\}) = \sup_{j \geq n} \rho(g - f_j)$. Therefore, $\rho(g - f) \leq \limsup_{n \to \infty} \rho(g - f_n)$ and (2) holds for any $f \in A$. We will prove that there exists $f \in A$ satisfying (1). Without loss of generality we may assume that $\delta(\{f_n\}_n) > 0$. Choose $\varepsilon > 0$ such that $\tilde{N}(L_\rho) \delta(\{f_n\}_n) + \varepsilon \leq c\delta(\{f_n\}_n)$. By definition of $R(A_m)$, there exists $g_n \in A_m$ such that $r(g_n, A_m) < R(A_m) + \varepsilon \leq \tilde{N}(L_\rho)\delta(A_m) + \varepsilon \leq \tilde{N}(L_\rho)\delta(\{f_n\}_n) + \varepsilon$. Hence, $\rho(f - g_n) \leq \rho(f - f_n) + \rho(f_n - g_n) \leq c\delta(\{f_n\}_n) + \varepsilon$.
\( \hat{N}(L_\rho)(\{f_n\}_n) + \varepsilon \leq c\delta(\{f_n\}_n). \) Since \( r(g_n,A_n) = r(g_n,\{f_j\}_{j \geq n}) = \sup_{j \geq n} \rho(g_n - f_j), \) we have

\[
\limsup_{j \to \infty} \rho(g_n - f_j) \leq c\delta(\{f_n\}_n).
\]

(A)

Using Lemma 2.5, there exists \( f \in \bigcap_{n=1}^\infty \text{ad}(g_i, i \geq n) \) such that

\[
\limsup_{j \to \infty} \rho(f - f_j) \leq \limsup_{n \to \infty} \limsup_{j \to \infty} \rho(g_n - f_j).
\]

(B)

We will check that \( f \in A. \) Indeed, for all \( i,n \) integers such that \( i \geq n \) we have \( g_i \in A_i \subset A_n. \) Thus, \( \{g_i\}_{i \geq n} \subset A_n \) which implies \( \text{ad}(g_i, i \geq n) \subset A_n \) and \( f \in A. \) Using (B) and (A) it is clear that \( \limsup_{j \to \infty} \rho(f - f_j) \leq c\delta(\{f_n\}_n). \)

**Theorem 2.1.** Let \( \rho \) be a convex function modular satisfying the \( \Delta_2 \)-condition and \( B \) an admissible, \( \rho \)-a.e. sequentially compact and \( \rho \)-bounded subset of \( L_\rho. \) Suppose that \( \hat{N}(L_\rho) < 1 \) and let \( T : B \to B \) be a \( k \)-uniformly Lipschitzian mapping satisfying \( k < (\hat{N}(L_\rho))^{-1/2}. \) Then, \( T \) has a fixed point.

**Proof.** We can assume that \( k > 1; \) otherwise \( T \) will be nonexpansive and the existence of a fixed point is a consequence of [8, Theorem 3.5]. Choose a constant \( c, \) \( \hat{N}(L_\rho) < c < 1 \) such that \( 1 < k < c^{-1/2}. \) Fix \( f_0 \in B. \) By Lemma 2.6, we can inductively construct a sequence \( \{f_j\}_{j \geq 0} \subset B \) such that for each \( j \geq 0 \)

1. \( \limsup_{n \to \infty} \rho(T^n(f_j) - f_{j+1}) \leq c\delta(\{T^n(f_j)\}_n). \)
2. \( \rho(f_{j+1} - g) \leq \limsup_{n \to \infty} \rho(T^n(f_j) - g) \) for all \( g \in B. \)

Denote \( D_j = \limsup_{n \to \infty} \rho(T^n(f_j) - f_{j+1}) \) and \( h = ck^2 < 1. \) For \( n \geq m \geq 0, \) we have

\[
\rho(T^m f_j - T^n f_j) \leq k\rho(f_j - T^{n-m} f_j)
\]

\[
\leq k \limsup_{i \to \infty} \rho(T^i f_{j-1} - T^{n-m} f_j)
\]

\[
\leq k^2 \limsup_{i \to \infty} \rho(T^{i-(n-m)} f_{j-1} - f_j)
\]

\[
\leq k^2 D_{j-1}.
\]

Since \( D_j = \limsup_{n \to \infty} \rho(T^n(f_j) - f_{j+1}) \leq c\delta(\{T^n(f_j)\}_n), \) we obtain \( D_j \leq ck^2 D_{j-1} = hD_{j-1}. \) Thus, \( D_j \leq h^j D_0 \) and we have

\[
\rho(f_{j+1} - f_j) \leq \omega(2)(\rho(f_{j+1} - T^n f_j) + \rho(f_j - T^n f_j))
\]

\[
\leq \omega(2)(\rho(f_{j+1} - T^n f_j) + \limsup_{m \to \infty} \rho(T^m f_{j-1} - T^n f_j))
\]

\[
\leq \omega(2)(\rho(f_{j+1} - T^n f_j) + k \limsup_{m \to \infty} \rho(T^{m-n} f_{j-1} - f_j))
\]

\[
\leq \omega(2)(\rho(f_{j+1} - T^n f_j) + kD_{j-1}).
\]

Taking \( \limsup \) as \( n \to \infty, \) we obtain

\[
\rho(f_{j+1} - f_j) \leq \omega(2)(D_j + kD_{j-1})
\]
\[ \leq \omega(2)(h_j + kh_j^{-1})D_0 \]
\[ \leq \omega(2)(h + k)h_j^{-1}D_0 \]
\[ \leq Ah_j, \text{ where } A = \omega(2)\frac{h + k}{h}D_0. \]

Hence, there exists an integer \( N \) and some \( \beta < 1 \) such that for \( j > N \) we have \( \rho(f^j_{j+1} - f_j) \leq \beta \), which implies \( 1/\beta \leq 1/\rho(f^j_{j+1} - f_j) \). Using properties (2) and (3) of Lemma 1.1 we obtain
\[ \omega^{-1} \left( \frac{1}{\beta} \right) \leq \omega^{-1} \left( \frac{1}{\rho(f^j_{j+1} - f_j)} \right) \]
and
\[ \left( \omega^{-1} \left( \frac{1}{\beta} \right) \right)^j \leq \omega^{-1} \left( \frac{1}{\rho(f^j_{j+1} - f_j)} \right)^j. \]

Therefore, by Lemma 2.2 we have
\[ \|f^j_{j+1} - f_j\|_\rho \leq \omega^{-1} \left( \frac{1}{\rho(f^j_{j+1} - f_j)} \right) \leq \omega^{-1} \left( \frac{1}{\rho(f^j_{j+1} - f_j)} \right)^j. \]

Hence \( \{f_j\} \) is a Cauchy sequence in \( (L_\rho, \| \cdot \|_\rho) \), there exists \( f \in L_\rho \) such that \( \|f_j - f\|_\rho \to 0 \), because \( (L_\rho, \| \cdot \|_\rho) \) is complete. Since under \( \Delta_2 \)-condition norm-convergence and modular-convergence are identical, \( \{f_j\} \) is modular convergent to \( f \). Thus, there exists a subsequence of \( \{f_j\} \) \( \rho \)-a.e. convergent to \( f \) [1, Theorem 1] and \( f \) belongs to \( B \) because \( B \) is \( \rho \)-a.e. sequentially closed. We will prove that \( f \) is a fixed point of \( T \). Indeed,
\[ \rho(f - Tf) \leq \omega(3)(\rho(f - f^j_{j+1}) + \rho(f^j_{j+1} - T^n f_j) + \rho(T^n f_j - Tf)) \]
\[ \leq \omega(3)(\rho(f - f^j_{j+1}) + \rho(f^j_{j+1} - T^n f_j) + k\rho(T^{n-1} f_j - f)) \]
\[ \leq \omega(3)g(\rho(f - f^j_{j+1}) + \rho(f^j_{j+1} - T^n f_j) + k\rho(f^j_{j+1} - T^n f_j) + k\omega(2)(\rho(T^{n-1} f_j - f^j_{j+1}) + \rho(f^j_{j+1} - f)))g). \]

Taking limsup as \( n \to \infty \), we have
\[ \rho(f - Tf) \leq \omega(3)(\rho(f - f^j_{j+1}) + D_j + k\omega(2)(D_j + \rho(f^j_{j+1} - f))). \]
Now, taking lim as \( j \to \infty \), we obtain \( \rho(f - Tf) = 0 \), i.e. \( T(f) = f \). \( \square \)

3. Uniformly convex modular spaces

Our goal in this section is to give some classes of modular functions spaces such that \( \tilde{N}(L_\rho) < 1 \). We begin by recalling the definitions of \( \rho \)-modulus of uniform convexity [9].
For any $\varepsilon$ and any $r > 0$, the $\rho$-modulus of uniform convexity is defined by

$$\delta_\rho(r, \varepsilon) = \inf \left\{ 1 - \frac{1}{r} \rho \left( \frac{f + g}{2} \right); \rho(f) \leq r, \rho(g) \leq r, \rho \left( \frac{f - g}{2} \right) \geq r \varepsilon \right\}$$

$$= \inf \left\{ 1 - \frac{1}{r} \rho \left( f + \frac{h}{2} \right); \rho(f) \leq r, \rho(f + h) \leq r; \rho \left( \frac{h}{2} \right) \geq r \varepsilon \right\}.$$

This following lemma gives a relationship between $\tilde{N}(L_\rho)$ and the $\rho$-modulus of uniform convexity.

**Lemma 3.1.** Let $\rho$ be a convex function modular satisfying $\Delta_2$-condition. Then,

$$\tilde{N}(L_\rho) \leq 1 - \inf_{d > 0} \delta_\rho(d, \gamma) \text{ for all } \gamma \in \left( 0, \frac{1}{\omega(2)} \right).$$

**Proof.** Let $B$ be an admissible, $\rho$-bounded and $\rho$-a.e. sequentially compact subset of $L_\rho$. We know that $B$ is a convex set because it is an intersection of $\rho$-balls which are convex, as a consequence of the convexity of $\rho$. Denote $d = \delta(B)$ and $r = R(B)$. Let $\varepsilon \in (0, 1)$. There exist $f, g \in B$ such that $\rho(f - g) \geq \epsilon \delta(B)$. Hence $\rho((f - g)/2) \geq \rho((f - g)/\omega(2)) \geq (d - \varepsilon)/\omega(2)$. Let $h \in B$. We know that $\rho(h - f) \leq d$, $\rho(h - g) \leq d$ and $\rho((h - f) - (h - g)/2) \geq d\varepsilon/\omega(2)$. By definition of $\delta_\rho(d, \varepsilon/\omega(2))$, we have

$$\rho \left( \frac{h - f + g}{2} \right) = \rho \left( \frac{(h - f) + (h - g)}{2} \right) \leq d \left( 1 - \delta_\rho \left( d, \frac{\varepsilon}{\omega(2)} \right) \right)$$

for all $h \in B$. Thus,

$$\frac{r}{d} \leq 1 - \delta_\rho \left( d, \frac{\varepsilon}{\omega(2)} \right).$$

Therefore,

$$\tilde{N}(L_\rho) \leq \sup_{d > 0} \left( 1 - \delta_\rho \left( d, \frac{\varepsilon}{\omega(2)} \right) \right)$$

$$\leq 1 - \inf_{d > 0} \delta_\rho \left( d, \frac{\varepsilon}{\omega(2)} \right). \quad \square$$

Let $\Phi : R \to R^+$ is said to be an $N$-function if $\Phi$ is a convex symmetric function which satisfies

1. $\Phi(0) = 0$
2. $\Phi$ is strictly increasing on $[0, \infty)$
3. $\lim_{u \to 0} \Phi(u)/u = 0$ and $\lim_{u \to \infty} \Phi(u)/u = \infty$.

Let $(G, \Sigma, \mu)$ be a measure space, $\mu$ being finite and atomless. Consider the space $L^0(G)$ consisting of all measurable real-valued functions on $G$, and define the Orlicz
If \( \delta \) satisfies the \( \Delta_2 \)-condition at zero and at infinity i.e. \( \limsup \), we obtain
\[
\rho(\lambda f) < \infty \text{ for some } \lambda > 0.
\]

If \( \Phi \) satisfies the \( \Delta_2 \)-condition at zero and at infinity i.e. \( \limsup \), \( \limsup_{u \to \infty} \Phi(2u)/\Phi(u) < \infty \) and \( \limsup_{u \to 0} \Phi(2u)/\Phi(u) < \infty \), the convex Orlicz modular associated to \( \Phi \) satisfies the \( \Delta_2 \)-type condition. We recall that the function \( \Phi \) is said to be uniformly convex if for all \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) \in (0, 1) \) such that
\[
0 \leq u \leq v \leq \varepsilon u \implies \Phi\left(\frac{u + v}{2}\right) \leq (1 - \delta(\varepsilon))\frac{\Phi(u) + \Phi(v)}{2}.
\]

(Some equivalent definitions can be found in [6].)

The following lemma connects the uniform convexity of \( \Phi \) and the \( \rho \)-modulus of uniform convexity of the modular.

**Lemma 3.2.** Let \( \Phi \) be a uniformly convex, N-function satisfying the \( \Delta_2 \)-condition at zero and at infinity and \( \rho \) the Orlicz function modular associated to \( \Phi \). Then there exists \( \varepsilon_0 \in (0, 1) \), such that for every \( \varepsilon \in (\varepsilon_0, 1) \) there exists \( \gamma(\varepsilon) \in (0, 1/\omega(2)) \) with \( \inf_{r > 0} \delta_\rho(r, \gamma(\varepsilon)) > 0 \).

**Proof.** We can find \( \varepsilon_0 \in (0, 1) \) such that \((1 - \varepsilon)/2\varepsilon < 1/\omega(2)\) for all \( \varepsilon \in (\varepsilon_0, 1) \), Choose \( \varepsilon \in (\varepsilon_0, 1) \). By definition of uniform convexity, there exists \( \delta(\varepsilon) \in (0, 1) \) such that
\[
0 \leq u \leq v \leq \varepsilon u \implies \Phi\left(\frac{u + v}{2}\right) \leq (1 - \delta(\varepsilon))\frac{\Phi(u) + \Phi(v)}{2}.
\]
Choose \( \gamma(\varepsilon) > 0 \) such that \((1 - \varepsilon)/2\varepsilon < \gamma(\varepsilon) < 1/\omega(2)\). Let \( r \) be a positive number and consider functions \( f, g \in L_\rho \) such that \( \rho(f) \leq r, \rho(f + g) \leq r \) and \( \rho(h/2) \geq \gamma(\varepsilon) \).

We consider the following sets:
\[
\begin{align*}
G_1 &= \{ t \in G / 0 \leq f(t), f(t) < \varepsilon(f(t) + h(t)) \}, \\
G_2 &= \{ t \in G / 0 \leq f(t), f(t) + h(t) < \varepsilon f(t) \}, \\
G_3 &= \{ t \in G / f(t) < 0, \varepsilon(f(t) + h(t)) \leq f(t) \}, \\
G_4 &= \{ t \in G / f(t) < 0, \varepsilon f(t) \leq f(t) + h(t) \}.
\end{align*}
\]

We have
\[
\rho\left(\frac{f + h}{2}\right) = \int_{G \setminus \bigcup_{i=1}^{4} G_i} \Phi\left(f(t) + \frac{h(t)}{2}\right) \, dt + \int_{\bigcup_{i=1}^{4} G_i} \Phi\left(f(t) + \frac{h(t)}{2}\right) \, dt.
\]
Using the definition of the uniform convexity for the function \( \Phi \) on \( G_1, G_2, G_3 \) and \( G_4 \) we obtain
\[
\Phi\left(\frac{f(t) + (f(t) + h(t))}{2}\right) \leq (1 - \delta(\varepsilon))\frac{\Phi(f(t)) + \Phi(f(t) + h(t))}{2}.
\]
for every \( t \in \bigcup_{i=1}^{4} G_i \). Hence, using the convexity of \( \Phi \) in \( G \setminus \bigcup_{i=1}^{4} G_i \) we have

\[
\int_G \Phi \left( f(t) + \frac{h(t)}{2} \right) \, dt \leq \int_{G \setminus \bigcup_{i=1}^{4} G_i} \Phi \left( f(t) + \frac{h(t)}{2} \right) \, dt
\]

\[
+ (1 - \delta(\varepsilon)) \int_{i=1}^{4} G_i \frac{\Phi(f(t)) + \Phi(f(t) + h(t))}{2} \, dt
\]

\[
\leq \int_{G \setminus \bigcup_{i=1}^{4} G_i} \frac{\Phi(f(t)) + \Phi(f(t) + h(t))}{2} \, dt
\]

\[
+ (1 - \delta(\varepsilon)) \int_{i=1}^{4} G_i \frac{\Phi(f(t)) + \Phi(f(t) + h(t))}{2} \, dt
\]

\[
= \int_G \frac{\Phi(f(t)) + \Phi(f(t) + h(t))}{2} \, dt
\]

\[
- \delta(\varepsilon) \int_{i=1}^{4} G_i \frac{\Phi(f(t)) + \Phi(f(t) + h(t))}{2} \, dt
\]

\[
\leq r - \delta(\varepsilon) \int_{i=1}^{4} G_i \Phi \left( \frac{h(t)}{2} \right) \, dt,
\]

where the last inequality is again a consequence of the convexity and symmetry of \( \Phi \), because

\[
\Phi \left( \frac{h(t)}{2} \right) = \Phi \left( \frac{(f(t) + h(t)) - f(t)}{2} \right)
\]

\[
\leq \frac{\Phi(f(t) + h(t)) + \Phi(-f(t))}{2}
\]

\[
= \frac{\Phi(f(t) + h(t)) + \Phi(f(t))}{2}.
\]

We claim that

\[
\Phi \left( \frac{h(t)}{2} \right) \leq \frac{1 - \varepsilon}{2\varepsilon} \Phi(f(t))
\]

for every \( t \in G \setminus \bigcup_{i=1}^{4} G_i \). To prove this inequality we will consider two cases:

**Case 1:** Assume \( f(t) \geq 0 \). Since \( t \in G \setminus \bigcup_{i=1}^{4} G_i \) we have

\[
-\frac{1 - \varepsilon}{2\varepsilon} f(t) \leq \frac{h(t)}{2} \leq \frac{1 - \varepsilon}{2\varepsilon} f(t).
\]

Therefore, by the symmetry and convexity of \( \Phi \), we obtain

\[
\Phi \left( \frac{h(t)}{2} \right) = \Phi \left( \left| \frac{h(t)}{2} \right| \right) \leq \Phi \left( \frac{1 - \varepsilon}{2\varepsilon} f(t) \right) \leq \frac{1 - \varepsilon}{2\varepsilon} \Phi(f(t)).
\]
Case 2: Assume $f(t) < 0$. Since $t \in G \setminus \bigcup_{i=1}^{i=4} G_i$ we have
\[ -\frac{\varepsilon - 1}{2\varepsilon} f(t) < \frac{h(t)}{2} < \frac{\varepsilon - 1}{2\varepsilon} f(t) \]
and we obtain
\[ \Phi \left( \frac{h(t)}{2} \right) = \Phi \left( \frac{|h(t)|}{2} \right) < \frac{1 - \varepsilon}{2\varepsilon} \Phi(f) \]
as above.

Thus we proved the inequality (II). Hence,
\[ \int_{G \setminus \bigcup_{i=1}^{i=4} G_i} \Phi \left( \frac{h(t)}{2} \right) \, dt \leq \frac{1 - \varepsilon}{2\varepsilon} \int_{G \setminus \bigcup_{i=1}^{i=4} G_i} \Phi(f(t)) \, dt \]
\[ \leq \frac{1 - \varepsilon}{2\varepsilon} r \]
and we obtain
\[ \int_{i=1}^{i=4} G_i \Phi \left( \frac{h(t)}{2} \right) \, dt = \int_{G} \Phi \left( \frac{h(t)}{2} \right) \, dt - \int_{G \setminus \bigcup_{i=1}^{i=4} G_i} \Phi \left( \frac{h(t)}{2} \right) \, dt \]
\[ \geq r\gamma(\varepsilon) - \frac{1 - \varepsilon}{2\varepsilon} r. \] (III)

From inequalities (I) and (III) we obtain
\[ \rho \left( f + \frac{h}{2} \right) \leq r \left( 1 - \delta(\varepsilon) \left( \gamma(\varepsilon) - \frac{1 - \varepsilon}{2\varepsilon} \right) \right) \]
and
\[ \delta(\varepsilon) \left( \gamma(\varepsilon) - \frac{1 - \varepsilon}{2\varepsilon} \right) \leq 1 - \frac{\rho \left( f + \frac{h}{2} \right)}{r} . \]

Thus
\[ \delta(\varepsilon) \left( \gamma(\varepsilon) - \frac{1 - \varepsilon}{2\varepsilon} \right) \leq \delta(r, \gamma(\varepsilon)) \quad \text{for every } r > 0 \]
and therefore
\[ \inf_{r > 0} \delta(r, \gamma(\varepsilon)) \geq \delta(\varepsilon) \left( \gamma(\varepsilon) - \frac{1 - \varepsilon}{2\varepsilon} \right) > 0. \]

Using Lemmas 3.1 and 3.2 we obtain the following corollary:

**Corollary 3.1.** Let $\Phi$ be a uniformly convex $N$-function satisfying the $\Delta_2$-condition at zero and at infinity. Then the modular function space $L_\rho$ associated to $\Phi$ satisfies $\tilde{N}(L_\rho) < 1$.

**Remark 3.1.** It is not difficult to find examples of functions satisfying the conditions in the above corollary. Besides $\Phi(t) = |t|^p$ for $p > 1$ we can obtain some other examples
using the following result [5]: \( \Phi \) is uniformly convex if \( \limsup_{t \to 0} \Phi'(at)/\Phi'(t) < 1 \) and \( \limsup_{t \to \infty} \Phi'(at)/\Phi'(t) < 1 \) for every \( a \in (0, 1) \). It is easy to check that \( \Phi(t) = t^2 - \log(1 + t^2) \) satisfies these conditions.

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