UNIFORMLY LIPSCHITZIAN MAPPINGS IN MODULAR FUNCTION SPACES

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ABSTRACT

Let ρ be a convex modular function satisfying a Δ_2-type condition and L_ρ the corresponding modular space. Assume that C is a ρ-bounded and ρ-a.e compact subset of L_ρ and $T : C \to C$ is a k-uniformly Lipschitzian mapping. We prove that T has a fixed point if $k < (\tilde{\mathrm{N}}(L_\rho))^{-1/2}$ where $\tilde{\mathrm{N}}(L_\rho)$ is a geometrical coefficient of normal structure. We also show that $\tilde{\mathrm{N}}(L_\rho) < 1$ in modular Orlicz spaces for uniformly convex Orlicz functions.

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INTRODUCTION

The theory of modular spaces was initiated by Nakano [14] in 1950 in connection with the theory of order spaces and redefined and generalized by Musielak and Orlicz [13] in 1959. Defining a norm, particular Banach spaces of functions can be considered. Metric fixed theory for these Banach spaces of functions has been widely studied (see, for instance, [15]). Another direction is based on considering an abstractly given functional which controls the growth of the functions. Even though a metric is not defined, many problems in fixed point theory for nonexpansive mappings can be reformulated in modular spaces (see, for instance, [8] and references therein). In this paper, we study the existence of fixed points for a more general class of mappings: uniformly Lipschitzian mappings. Fixed point theorems for this class of mappings in Banach spaces have been studied in [3,4] and in metric spaces in [11,12] (for further information about this subject, see [2, chapter VIII] and references therein). The main tool in our approach is the coefficient of normal structure $\tilde{\mathrm{N}}(L_\rho)$. We prove that under suitable conditions a k-uniformly Lipschitzian mapping has a fixed point if $k < (\tilde{\mathrm{N}}(L_\rho))^{-1/2}$. In the last section we show a class of modular spaces where $\tilde{\mathrm{N}}(L_\rho) < 1$ and so, the above theorem can be successfully applied.

1. PRELIMINARIES

We start by recording a brief collection of basic concepts and facts of modular spaces as formulated by Kozlowski. For more details the reader is refered to [7], [8], [10] and [13].

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Σ, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. In other words, the family $\mathcal{P}$ plays the role of the δ-ring of subsets of finite measure. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}$ we will denote the space of all measurable functions, i.e. all functions $f : \Omega \to \Re$ such that there exists a sequence $\{g_n\} \in \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 1.1. A functional $\rho : \mathcal{E} \times \Sigma \to [0,\infty]$ is called a function modular if

(P_1) $\rho(0,E) = 0$ for any $E \in \Sigma$,

(P_2) $\rho(f,E) \le \rho(g,E)$ whenever $|f(\omega)| \le |g(\omega)|$ for any $\omega \in \Omega$, $f,g \in \mathcal{E}$ and $E \in \Sigma$,

(P_3) $\rho(f,.) : \Sigma \to [0,\infty]$ is a σ-subadditive measure for every $f \in \mathcal{E}$,

(P_4) $\rho(\alpha,A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha,A) = \rho(\alpha 1_A, A)$,

(P_5) if there exists $\alpha > 0$ such that $\rho(\alpha,A) = 0$, then $\rho(\beta,A) = 0$ for every $\beta > 0$,

(P_6) for any $\alpha > 0$ $\rho(\alpha,.)$ is order continuous on $\mathcal{P}$, that is $\rho(\alpha,A_n) \to 0$ if $\{A_n\} \in \mathcal{P}$ and decreases to $\emptyset$.

The definition of ρ is then extended to $f \in \mathcal{M}$ by

$$\rho(f,E) = \sup\{\rho(g,E); g \in \mathcal{E},\ |g(\omega)| \le |f(\omega)|\ \ \omega \in \Omega\}.$$

A set E is said to be ρ-null if $\rho(\alpha,E) = 0$ for every $\alpha > 0$. For the sake of simplicity we write $\rho(f)$ instead of $\rho(f,\Omega)$.
It is easy to see that the functional $\rho : \mathcal{M} \to [0,\infty]$ is a modular because it satisfies the following properties:

(i) $\rho(f) = 0$ iff $f = 0$ ρ-a.e.

(ii) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.

(iii) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha \ge 0, \beta \ge 0$ and $f,g \in \mathcal{M}$.

In addition, if the following property is satisfied

(iii)' $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$; $\alpha \ge 0, \beta \ge 0$ and $f,g \in \mathcal{M}$,

we say that ρ is a convex modular.

The modular ρ defines a corresponding modular space, i.e the vector space L_ρ given by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \ \text{ as } \ \lambda \to 0\}.$$

We can also consider the space $E_\rho = \{f \in \mathcal{M}; \rho(\alpha f, A_n) \to 0 \quad \text{as} \quad n \to \infty \ \text{ for every } \ A_n \in \Sigma \ \text{ that decreases to } \ \emptyset \ \text{ and } \ \alpha > 0\}$.

A function modular is said to satisfy the Δ_2-condition if $\sup_{n\geq 1} \rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{f_n\}_{n\geq 1} \subset \mathcal{M}$, $D_k \in \Sigma$ decreases to $\emptyset$ and $\sup_{n\geq 1} \rho(f_n, D_k) \to 0$ as $k \to \infty$. We know (see [10]) that $E_\rho = L_\rho$ when ρ satisfies the Δ_2-condition. When ρ is convex, the formula

$$||f||_\rho = \inf \left\{ \alpha > 0; \rho\left(\frac{f}{\alpha}\right) \leq 1 \right\}$$

definies a norm in the modular space L_ρ which is frequently called the Luxemburg norm.

Definition 1.2.

(1) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$,
(2) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-a.e convergent to $f \in L_\rho$ if the set $\{\omega \in \Omega; f_n(\omega) \not\to f(\omega)\}$ is ρ-null.
(3) The sequence $\{f_n\}_n \subset L_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m go to ∞,
(4) A subset C of L_ρ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.
(5) A subset C of L_ρ is called ρ-a.e sequentially closed if the ρ-a.e limit of a ρ-a.e convergent sequence of C always belongs to C.
(6) A subset C of L_ρ is called ρ-a.e sequentially compact if every sequence in C has a ρ-a.e convergent subsequence in C.
(7) A subset C of L_ρ is called ρ-bounded if

$$\delta_\rho(C) = \sup\{\rho(f - g); f, g \in C\} < \infty.$$

Let B be a bounded subset of L_ρ. We define the ρ-ball of center $f \in L_\rho$ and radius $r > 0$ by $B(f, r) = \{g \in L_\rho, \ \ \rho(g - f) \leq r\}$. We will denote $r(f, B) = \sup\{\rho(f - g), \ \ g \in B\}$, $\delta(B) = \sup\{r(f, B), \ \ f \in B\}$, $R(B) = \inf\{r(f, B), \ \ f \in B\}$. We define the admissible hull of B as the intersection of all ρ-ball containing B, i.e:

$$ad(B) = \bigcap\{A : B \subset A \subset L_\rho, \ \ \text{where } A \text{ is a } \rho\text{-ball}\}.$$

B is said admissible if $ad(B) = B$. We define the normal structure coefficient $\tilde{\mathrm{N}}(L_\rho)$ of L_ρ by

$$\tilde{\mathrm{N}}(L_\rho) = \sup\left\{\frac{R(B)}{\delta(B)}\ ,\quad B \text{ is admissible, } \rho\text{-bounded and } \rho\text{-a.e sequentially compact}\right\}.$$

The useful following proposition is easily seen:

Proposition 1.1. *Let B be a ρ-bounded subset of L_ρ and $f \in L_\rho$. Then*

(1) $r(f, ad(B)) = r(f, B)$.
(2) $\delta(ad(B)) = \delta(B)$.

We say that ρ satisfies the Δ_2-type condition if there exists $K > 0$ such that $\rho(2f) \leq K\rho(f)$ for all $f \in L_\rho$. In general, Δ_2-type condition and Δ_2-condition are not equivalent, even though it is obvious that Δ_2-type condition implies Δ_2-condition. Assume that ρ is convex and satisfies the Δ_2-type condition. We define a growth function ω by

$$\omega(t) = \sup\left\{\frac{\rho(tf)}{\rho(f)},\ \ 0 < \rho(f) < \infty\right\} \quad \text{for all} \quad 0 \leq t < \infty.$$

The following properties of the growth function can be easily seen.

Lemma 1.1. *Let ρ be a convex function modular satisfying the Δ_2-type condition. Then the growth funtion ω has the following properties:*

(1) $\omega(t) < \infty\ ,\forall t \in [0, \infty)$

(2) $\omega : [0, \infty) \to [0, \infty)$ *is a convex, strictly increasing function. So, it is continuous.*

(3) $\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta); \forall \alpha, \beta \in [0, \infty)$

(4) $\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta)$;$\forall \alpha, \beta \in [0, \infty)$, *where ω^{-1} is the function inverse of ω.*

The following lemma shows that the growth function can be used to give an upper bound for the norm of a function.

Lemma 1.2. [5] *Let ρ be a convex function modular satisfying the Δ_2-type condition. Then*

$$||f||_\rho \leq \frac{1}{\omega^{-1}\left(\frac{1}{\rho(f)}\right)} \quad \textit{whenever} \quad f \in L_\rho.$$

The following lemma can be found in [7].

Lemma 1.3. *Let ρ be a function modular satisfying the Δ_2-condition and $\{f_n\}_n$ be a sequence in L_ρ such that $f_n \overset{\rho-a.e}{\rightarrow} f \in L_\rho$ and there exists $k > 1$ such that $\sup_n \rho(k(f_n - f)) < \infty$. Then,*

$$\liminf_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_n - f) + \rho(f - g) \quad \textit{for all} \;\; g \in L_\rho.$$

Lemma 1.4. *Let ρ be a modular function satisfying the Δ_2-type condition. Let B be a ρ-a.e sequentially closed and ρ-bounded subset of L_ρ. Let $\{g_n\}_n$ be a sequence in B such that $g_n \overset{\rho-a.e}{\rightarrow} g$. Then,*

(1) $\rho(g) \leq \liminf_{n\to\infty} \rho(g_n)$.
(2) $B(0, r) \cap B$ *is ρ-a.e sequentially closed.*
(3) $ad(A) \cap B$ *is ρ-a.e sequentially closed, for all $A \subset L_\rho$.*

Proof. Condition (1) is a straighforward consequence of Lemma 1.3 applied to the sequence $g_n \overset{\rho-a.e}{\rightarrow} g$ and the null function. Condition (2) and (3) can be easily deduced from (1).

2. FIXED POINT FOR UNIFORMLY LIPSCHITZIAN MAPPINGS

The following lemma is the key of our fixed point result.

Lemma 2.1. *Let ρ be a modular function satisfying the Δ_2-type condition and B a ρ-bounded and ρ-a.e sequentially compact subset of L_ρ. Let $\{f_n\}_n$ and $\{g_n\}_n$ be sequences in B. Then, there exists $g \in \cap_{n=1}^{\infty} ad(g_j,\ j \geq n) \cap B$ such that*

$$\limsup_{n\to\infty} \rho(g - f_n) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} \rho(g_j - f_n)$$

Proof. Let $\{f_n\}_n$ and $\{g_n\}_n$ be sequences in B. We define $\theta(h) = \limsup_{n\to\infty} \rho(h - f_n)$ for all $h \in B$. Since B is ρ-sequentially compact and ρ-bounded, there exist a subsequence $\{g_{\phi(n)}\}_n \subset \{g_n\}_n$ such that $g_{\phi(n)} \overset{\rho-a.e}{\rightarrow} g$ and a subsequence

$\{f_{\psi(n)}\}_n \subset \{f_n\}_n$ such that $\lim_{n\to\infty} \rho(f_{\psi(n)} - g) = \limsup_{n\to\infty} \rho(f_n - g)$ and $f_{\psi(n)} \overset{\rho-a.e}{\to} f \in B$. Since $g_{\phi(n)} \in ad(g_j\ ,\ j \geq n) \cap B$ which is ρ-a.e sequentially closed (by property (3) of Lemma 1.4) and $g_{\phi(n)} \overset{\rho-a.e}{\to} g$, we obtain $g \in ad(g_j\ ,\ j \geq n) \cap B$ for all $n \geq 1$. We will see that $\theta(g) \leq \limsup_{j\to\infty} \theta(g_j)$. Indeed, from Lemma 2.3 we have $\theta(g_j) = \limsup_{n\to\infty} \rho(f_n - g_j) \geq \liminf_{n\to\infty} \rho(f_{\psi(n)} - g_j) = \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \rho(f - g_j)$. Thus, again using Lemma 2.3, we obtain

$$\begin{aligned}\limsup_{j\to\infty} \theta(g_j) &\geq \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \limsup_{j\to\infty} \rho(f - g_j) \\ &\geq \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \liminf_{j\to\infty} \rho(f - g_{\phi(j)}) \\ &= \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \liminf_{j\to\infty} \rho(g_{\phi(j)} - g) + \rho(f - g).\end{aligned}$$

On the other hand $\theta(g) = \limsup_{n\to\infty} \rho(f_n - g) = \liminf_{n\to\infty} \rho(f_{\psi(n)} - g) = \liminf_{n\to\infty} \rho(f_{\psi(n)} - f) + \rho(f - g)$. Therefore, $\theta(g) \leq \limsup_{j\to\infty} \theta(g_j)$.

The following lemma is inspired on [3] where a similar lemma is proved in reflexive Banach spaces (see also [12, Lemma 6] for a version in metric spaces with additional properties).

Lemma 2.2. *Let ρ be a function modular satisfying the Δ_2-type condition and B an be admissible, ρ-a.e sequentially compact and ρ-bounded subset of L_ρ. Let $\{f_n\}$ be a sequence in B and c a constant such that $c > \tilde{N}(L_\rho)$. Then there exists $f \in B$ such that*

(1) $\limsup_{n\to\infty} \rho(f - f_n) \leq c\, \delta(\{f_n\}_n)$.

(2) $\rho(f - g) \leq \limsup_{n\to\infty} \rho(f_n - g)$ *for all* $g \in B$.

Proof. Let $\{f_n\}_n$ be a sequence of B. Denote $A_m = ad(f_j : j \geq m) \subset B$ and $A = \bigcap_{m=1}^{\infty} A_m$. Since B is ρ-a.e sequentially compact, there exists a subsequence of $\{f_n\}_n$ $\rho - a.e$ convergent, say to h. It is clear that $h \in A$ and so $A \neq \emptyset$. Furthermore, from Proposition 1.1 (2), we have $\delta(A_n) \leq \delta(\{f_n\}_n)$. On the other hand, for any $f \in A$ and $g \in B$ we have $\rho(g-f) \leq r(g, A) \leq r(g, A_n) = r(g, \{f_j : j \geq n\}) = \sup_{j\geq n} \rho(g - f_j)$. Therefore, $\rho(g - f) \leq \limsup_{n\to\infty} \rho(g - f_n)$ and (2) holds for any $f \in A$. We will prove that there exists $f \in A$ satisfying (1). Without loss of generality we may assume that $\delta(\{f_n\}_n) > 0$. Choose $\varepsilon > 0$ such that $\tilde{\text{N}}(L_\rho)\delta(\{f_n\}_n)+\varepsilon \leq c\,\delta(\{f_n\}_n)$. By definition of $R(A_n)$, there exists $g_n \in A_n$ such that $r(g_n, A_n) < R(A_n)+\varepsilon \leq \tilde{\text{N}}(L_\rho)\delta(A_n)+\varepsilon \leq \tilde{\text{N}}(L_\rho)\delta(\{f_n\}_n)+\varepsilon \leq c\,\delta(\{f_n\}_n)$. Since $r(g_n, A_n) = r(g_n, \{f_j\}_{j\geq n}) = \sup_{j\geq n} \rho(g_n - f_j)$, we have,

$$\limsup_{j\to\infty} \rho(g_n - f_j) \leq c\,\delta(\{f_n\}_n) \qquad (A).$$

Using Lemma 2.5, there exists $f \in \cap_{n=1}^{\infty} ad(g_i,\ i \geq n)$ such that

$$\limsup_{j\to\infty} \rho(f - f_j) \leq \limsup_{n\to\infty} \limsup_{j\to\infty} \rho(g_n - f_j) \qquad (B).$$

We will check that $f \in A$. Indeed, for all i, n integers such that $i \geq n$ we have $g_i \in A_i \subset A_n$. Thus, $\{g_i\}_{i\geq n} \subset A_n$ which implies $ad(g_i,\ i \geq n) \subset A_n$ and $f \in A$. Using (B) and (A) it is clear that $\limsup_{j\to\infty} \rho(f - f_j) \leq c\,\delta(\{f_n\}_n)$.

Theorem 2.1. *Let ρ be a convex function modular satisfying the Δ_2-condition and B an admissible, ρ-a.e sequentially compact and ρ-bounded subset of L_ρ. Suppose that $\tilde{N}(L_\rho) < 1$ and let $T : B \to B$ be a k-uniformly lipschitzian mapping satisfying $k < (\tilde{N}(L_\rho))^{-1/2}$. Then, T has a fixed point.*

Proof. We can assume that $k > 1$; otherwise T will be nonexpansive and the existence of a fixed point is a consequence of [8, Theorem 3.5]. Choose a constant c, $\tilde{\text{N}}(L_\rho) < c < 1$ such that $1 < k < c^{-1/2}$. Fix $f_0 \in B$. By Lemma 2.6, we can inductively construct a sequence $\{f_j\}_{j\geq 0} \subset B$ such that for each $j \geq 0$

(1) $\limsup_{n\to\infty} \rho(T^n(f_j) - f_{j+1}) \leq c\,\delta(\{T^n(f_j)\}_n)$.
(2) $\rho(f_{j+1} - g) \leq \limsup_{n\to\infty} \rho(T^n(f_j) - g)$ for all $g \in B$.

Denote $D_j = \limsup_{n\to\infty} \rho(T^n(f_j) - f_{j+1})$ and $h = ck^2 < 1$. For $n \geq m \geq 0$, we have

$$\begin{aligned}\rho(T^m f_j - T^n f_j) &\leq k\rho(f_j - T^{n-m} f_j)\\ &\leq k \limsup_{i\to\infty} \rho(T^i f_{j-1} - T^{n-m} f_j)\\ &\leq k^2 \limsup_{i\to\infty} \rho(T^{i-(n-m)} f_{j-1} - f_j)\\ &\leq k^2 D_{j-1}.\end{aligned}$$

Since $D_j = \limsup_{n\to\infty} \rho(T^n(f_j) - f_{j+1}) \leq c\ \delta(\{T^n(f_j)\}_n)$, we obtain $D_j \leq c\, k^2\ D_{j-1} = hD_{j-1}$. Thus, $D_j \leq h^j\ D_0$ and we have

$$\begin{aligned}\rho(f_{j+1} - f_j) &\leq \omega(2)\big(\rho(f_{j+1} - T^n f_j) + \rho(f_j - T^n f_j)\big)\\ &\leq \omega(2)\big(\rho(f_{j+1} - T^n f_j) + \limsup_{m\to\infty} \rho(T^m f_{j-1} - T^n f_j)\big)\\ &\leq \omega(2)\big(\rho(f_{j+1} - T^n f_j) + k \limsup_{m\to\infty} \rho(T^{m-n} f_{j-1} - f_j)\big)\\ &\leq \omega(2)\big(\rho(f_{j+1} - T^n f_j) + kD_{j-1}\big).\end{aligned}$$

Taking limsup as $n \to \infty$, we obtain

$$\begin{aligned}\rho(f_{j+1} - f_j) &\leq \omega(2)(D_j + kD_{j-1})\\ &\leq \omega(2)(h^j + kh^{j-1})D_0\\ &\leq \omega(2)(h+k)h^{j-1}D_0\\ &\leq Ah^j, \quad \text{where } A = \omega(2)\frac{h+k}{h}D_0.\end{aligned}$$

Hence, there exists an integer N and some $\beta < 1$ such that for $j > N$ we have $\rho(f_{j+1} - f_j) \leq \beta^j$, which implies $\dfrac{1}{\beta^j} \leq \dfrac{1}{\rho(f_{j+1} - f_j)}$. Using properties (2) and (3) of Lemma 1.1 we obtain

$$\omega^{-1}\left(\frac{1}{\beta^j}\right) \leq \omega^{-1}\left(\frac{1}{\rho(f_{j+1} - f_j)}\right)$$

and

$$\left(\omega^{-1}\left(\frac{1}{\beta}\right)\right)^j \leq \omega^{-1}\left(\frac{1}{\rho(f_{j+1} - f_j)}\right).$$

Therefore, by Lemma 2.2 we have

$$||f_{j+1} - f_j||_\rho \leq \frac{1}{\omega^{-1}\left(\frac{1}{\rho(f_{j+1}-f_j)}\right)} \leq \frac{1}{\left(\omega^{-1}(\frac{1}{\beta})\right)^j}.$$

Hence $\{f_j\}$ is a Cauchy sequence in $(L_\rho, ||.||_\rho)$, there exists $f \in L_\rho$ such that $||f_j - f||_\rho \to 0$, because $(L_\rho, ||.||_\rho)$ is complete. Since under Δ_2-condition norm-convergence and modular-convergence are identical, $\{f_j\}$ is modular convergent to f. Thus, there exists a subsequence of $\{f_j\}_j$ ρ-a.e convergent to f [1, Theorem 1] and f belongs to B because B is ρ-a.e sequentially closed. We will prove that f is a fixed point of T. Indeed,

$$\begin{aligned}
\rho(f - Tf) &\leq \omega(3)\big(\rho(f - f_{j+1}) + \rho(f_{j+1} - T^n f_j) + \rho(T^n f_j - Tf)\big)\\
&\leq \omega(3)\big(\rho(f - f_{j+1}) + \rho(f_{j+1} - T^n f_j) + k\rho(T^{n-1} f_j - f)\big)\\
&\leq \omega(3)\Bigg(\rho(f - f_{j+1}) + \rho(f_{j+1} - T^n f_j)\\
&\qquad + k\omega(2)\big(\rho(T^{n-1} f_j - f_{j+1}) + \rho(f_{j+1} - f)\big)\Bigg).
\end{aligned}$$

Taking limsup as $n \to \infty$, we have

$$\rho(f - Tf) \leq \omega(3)\Bigg(\rho(f - f_{j+1}) + D_j + k\omega(2)\big(D_j + \rho(f_{j+1} - f)\big)\Bigg).$$

Now, taking lim as $j \to \infty$, we obtain $\rho(f - Tf) = 0$, i.e. $T(f) = f$.

3. UNIFORMLY CONVEX MODULAR SPACES

Our goal in this section is to give some classes of modular funtions spaces such that $\tilde{\mathrm{N}}(L_\rho) < 1$. We begin by recalling the definitions of ρ-modulus of uniform convexity [9].

For any ε and any $r > 0$, the ρ-modulus of uniform convexity is defined by

$$\begin{aligned}
\delta_\rho(r, \varepsilon) &= \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) \ ; \ \rho(f) \leq r, \rho(g) \leq r, \rho\left(\frac{f-g}{2}\right) \geq r\varepsilon\right\}.\\
&= \inf\left\{1 - \frac{1}{r}\rho\left(f + \frac{h}{2}\right) \ ; \ \rho(f) \leq r, \rho(f+h) \leq r; \rho\left(\frac{h}{2}\right) \geq r\varepsilon\right\}.
\end{aligned}$$

This following lemma gives a relationship between $\tilde{\mathrm{N}}(L_\rho)$ and the ρ-modulus of uniform convexity.

Lemma 3.1. *Let ρ be a convex function modular satisfying Δ_2-condition. Then,*

$$\tilde{N}(L_\rho) \leq 1 - \inf_{d>0} \delta_\rho(d,\gamma) \quad \textit{for all} \quad \gamma \in \left(0, \frac{1}{\omega(2)}\right).$$

Proof. Let B be an admissible, ρ-bounded and ρ-a.e sequentially compact subset of L_ρ. We know that B is a convex set because it is an intersection of ρ-balls which are convex, as a consequence of the convexity of ρ. Denote $d = \delta(B)$ and $r = R(B)$. Let $\varepsilon \in (0,1)$. There exist $f, g \in B$ such that $\rho(f-g) \geq \varepsilon\delta(B)$. Hence $\rho\left(\frac{f-g}{2}\right) \geq \frac{\rho(f-g)}{\omega(2)} \geq \frac{d-\varepsilon}{\omega(2)}$. Let $h \in B$. We know that $\rho(h-f) \leq d$, $\rho(h-g) \leq d$ and $\rho\left(\frac{(h-f)-(h-g)}{2}\right) \geq \frac{d\varepsilon}{\omega(2)}$. By definition of $\delta_\rho\left(d, \frac{\varepsilon}{\omega(2)}\right)$, we have

$$\begin{aligned} \rho\left(h - \frac{f+g}{2}\right) &= \rho\left(\frac{(h-f)+(h-g)}{2}\right) \\ &\leq d\left(1 - \delta_\rho\left(d, \frac{\varepsilon}{\omega(2)}\right)\right), \end{aligned}$$

for all $h \in B$. Thus,

$$\frac{r}{d} \leq 1 - \delta_\rho\left(d, \frac{\varepsilon}{\omega(2)}\right).$$

Therefore,

$$\begin{aligned} \tilde{\mathrm{N}}(L_\rho) &\leq \sup_{d>0}\left(1 - \delta_\rho\left(d, \frac{\varepsilon}{\omega(2)}\right)\right) \\ &\leq 1 - \inf_{d>0} \delta_\rho\left(d, \frac{\varepsilon}{\omega(2)}\right). \end{aligned}$$

Let $\Phi : R \to R^+$ is said to be an N-function if Φ is a convex symmetric function which satisfies:

(1) $\Phi(0) = 0$
(2) Φ is strictly increasing on $[0, \infty)$

(3) $\lim_{u\to 0} \dfrac{\Phi(u)}{u} = 0$ and $\lim_{u\to\infty} \dfrac{\Phi(u)}{u} = \infty$.

Let (G, Σ, μ) be a measure space, μ being finite and atomless. Consider the space $L^0(G)$ consisting of all measurable real-valued functions on G, and define the Orlicz function modular $\rho(f, B) = \int_{t\in B} \Phi(f(t))d\mu(t)$ for every $f \in L^0(G)$ and $B \in \Sigma$. The modular function space L_ρ is the Orlicz space defined by

$$L_\rho = \left\{f \in L^0(G),\ \ \rho(\lambda f) < \infty \ \text{ for some } \ \lambda > 0\right\}.$$

If Φ satisfies the Δ_2-condition at zero and at infinity i.e: $\limsup_{u\to 0} \dfrac{\Phi(2u)}{\Phi(u)} < \infty$ and $\limsup_{u\to\infty} \dfrac{\Phi(2u)}{\Phi(u)} < \infty$, the convex Orlicz modular associated to Φ satisfies the Δ_2-type condition. We recall that the function Φ is said to be uniformly convex [8] if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, 1)$ such that:

$$0 \le u \ \text{ and } \ v \le \varepsilon\, u \ \text{ implies } \ \Phi\left(\frac{u+v}{2}\right) \le (1 - \delta(\varepsilon))\frac{\Phi(u) + \Phi(v)}{2}.$$

(Some equivalent definitions can be found in [1]).
The following lemma connects the uniform convexity of Φ and the ρ-modulus of uniform convexity of the modular.

Lemma 3.2. *Let Φ be a uniformly convex, N-function satisfying the Δ_2-condition at zero and at infinity and ρ the Orlicz function modular associated to Φ. Then there exists $\varepsilon_0 \in (0, 1)$, such that for every $\varepsilon \in (\varepsilon_0, 1)$ there exists $\gamma(\varepsilon) \in \left(0, \frac{1}{\omega(2)}\right)$ with $\inf_{r>0} \delta_\rho\left(r, \gamma(\varepsilon)\right) > 0$.*

Proof. We can find $\varepsilon_0 \in (0, 1)$ such that $\dfrac{1-\varepsilon}{2\varepsilon} < \dfrac{1}{\omega(2)}$ for all $\varepsilon \in (\varepsilon_0, 1)$, Choose $\varepsilon \in (\varepsilon_0, 1)$. By definition of uniform convexity, there exists $\delta(\varepsilon) \in (0, 1)$ such that

$$0 \le u \ \text{ and } \ v \le \varepsilon\, u \ \text{ implies } \ \Phi\left(\frac{u+v}{2}\right) \le (1 - \delta(\varepsilon))\frac{\Phi(u) + \Phi(v)}{2}.$$

Choose $\gamma(\varepsilon) > 0$ such that $\dfrac{1-\varepsilon}{2\varepsilon} < \gamma(\varepsilon) < \dfrac{1}{\omega(2)}$. Let r be a positive number and consider functions $f, g \in L_\rho$ such that $\rho(f) \leq r$, $\rho(f+g) \leq r$ and $\rho\left(\dfrac{h}{2}\right) \geq r\gamma(\varepsilon)$.
We consider the following sets

$$\begin{aligned} G_1 &= \{t \in G \,/\, 0 \leq f(t) \,,\; f(t) < \varepsilon(f(t)+h(t))\}\,. \\ G_2 &= \{t \in G \,/\, 0 \leq f(t) \,,\; f(t)+h(t) < \varepsilon f(t)\}\,. \\ G_3 &= \{t \in G \,/\, f(t) < 0 \,,\; \varepsilon(f(t)+h(t)) \leq f(t)\}\,. \\ G_4 &= \{t \in G \,/\, f(t) < 0 \,,\; \varepsilon f(t) \leq f(t)+h(t)\}\,. \end{aligned}$$

We have

$$\rho\left(f+\frac{h}{2}\right) = \int_{G\setminus\bigcup_{i=1}^{i=4} G_i} \Phi\left(f(t)+\frac{h(t)}{2}\right) dt + \int_{\bigcup_{i=1}^{i=4} G_i} \Phi\left(f(t)+\frac{h(t)}{2}\right) dt.$$

Using the definition of the uniform convexity for the function Φ on G_1, G_2, G_3 and G_4 we obtain,

$$\Phi\left(\frac{f(t)+(f(t)+h(t))}{2}\right) \leq (1-\delta(\varepsilon))\,\frac{\Phi(f(t))+\Phi(f(t)+h(t))}{2}$$

for every $t \in \cup_{i=1}^{i=4} G_i$. Hence, using the convexity of Φ in $G \setminus \bigcup_{i=1}^{i=4} G_i$ we have

$$\begin{aligned}
\int_G \Phi\left(f(t) + \frac{h(t)}{2}\right) dt &\leq \int_{G \setminus \bigcup_{i=1}^{i=4} G_i} \Phi\left(f(t) + \frac{h(t)}{2}\right) dt + \\
&\quad (1-\delta(\varepsilon)) \int_{\bigcup_{i=1}^{i=4} G_i} \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} dt \\
&\leq \int_{G \setminus \bigcup_{i=1}^{i=4} G_i} \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} dt + \\
&\quad (1-\delta(\varepsilon)) \int_{\bigcup_{i=1}^{i=4} G_i} \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} dt \\
&= \int_G \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} dt - \\
&\quad \delta(\varepsilon) \int_{\bigcup_{i=1}^{i=4} G_i} \frac{\Phi(f(t)) + \Phi(f(t)+h(t))}{2} dt \\
&\leq r - \delta(\varepsilon) \int_{\bigcup_{i=1}^{i=4} G_i} \Phi\left(\frac{h(t)}{2}\right) dt, \qquad \text{(I)}
\end{aligned}$$

where the last inequality is again a consequence of the convexity and symmetry of Φ, because

$$\begin{aligned}
\Phi\left(\frac{h(t)}{2}\right) &= \Phi\left(\frac{(f(t)+h(t)) - f(t)}{2}\right) \\
&\leq \frac{\Phi(f(t)+h(t)) + \Phi(-f(t))}{2} \\
&= \frac{\Phi(f(t)+h(t)) + \Phi(f(t))}{2}.
\end{aligned}$$

We claim that

$$\Phi\left(\frac{h(t)}{2}\right) \leq \frac{1-\varepsilon}{2\varepsilon} \Phi(f(t)) \qquad \text{(II)}$$

for every $t \in G \setminus \cup_{i=1}^{i=4} G_i$. To prove this inequality we will consider two cases:
First case Assume $f(t) \geq 0$. Since $t \in G \setminus \cup_{i=1}^{i=4} G_i$ we have

$$-\frac{1-\varepsilon}{2\varepsilon} f(t) \leq \frac{h(t)}{2} \leq \frac{1-\varepsilon}{2\varepsilon} f(t).$$

Therefore, by the symmetry and convexity of Φ, we obtain

$$\Phi\left(\frac{h(t)}{2}\right) = \Phi\left(\left|\frac{h(t)}{2}\right|\right) \le \Phi\left(\frac{1-\varepsilon}{2\varepsilon} f(t)\right) \le \frac{1-\varepsilon}{2\varepsilon}\Phi(f(t)).$$

Second case Assume $f(t) < 0$. Since $t \in G \setminus \cup_{i=1}^{i=4} G_i$ we have

$$-\frac{\varepsilon-1}{2\varepsilon} f(t) < \frac{h(t)}{2} < \frac{\varepsilon-1}{2\varepsilon} f(t)$$

and we obtain

$$\Phi\left(\frac{h(t)}{2}\right) = \Phi\left(\left|\frac{h(t)}{2}\right|\right) < \frac{1-\varepsilon}{2\varepsilon}\Phi(f)$$

as above.
Thus we proved the inequality (II). Hence,

$$\begin{aligned}\int_{G\setminus\bigcup_{i=1}^{i=4} G_i} \Phi\left(\frac{h(t)}{2}\right) dt &\le \frac{1-\varepsilon}{2\varepsilon} \int_{G\setminus\bigcup_{i=1}^{i=4} G_i} \Phi(f(t))dt \\ &\le \frac{1-\varepsilon}{2\varepsilon} r\end{aligned}$$

and we obtain

$$\begin{aligned}\int_{\bigcup_{i=1}^{i=4} G_i} \Phi\left(\frac{h(t)}{2}\right) dt &= \int_G \Phi\left(\frac{h(t)}{2}\right) dt - \int_{G\setminus\bigcup_{i=1}^{i=4} G_i} \Phi\left(\frac{h(t)}{2}\right) dt \\ &\ge r\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon} r \qquad \text{(III)}.\end{aligned}$$

From inequalities (I) and (III) we obtain

$$\rho(f + \frac{h}{2}) \le r\left(1 - \delta(\varepsilon)\left(\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon}\right)\right)$$

and

$$\delta(\varepsilon)\left(\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon}\right) \le 1 - \frac{\rho\left(f + \frac{h}{2}\right)}{r}.$$

Thus

$$\delta(\varepsilon)\left(\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon}\right) \le \delta_\rho(r, \gamma(\varepsilon)) \quad \text{for every} \quad r > 0$$

and therefore

$$\inf_{r>0} \delta_\rho(r, \gamma(\varepsilon)) \geq \delta(\varepsilon) \left(\gamma(\varepsilon) - \frac{1-\varepsilon}{2\varepsilon} \right) > 0.$$

Using Lemma 3.1 and Lemma 3.2 we obtain the following corollary:

Corollary 3.1 *Let Φ be a uniformly convex N-function satisfying the Δ_2-condition at zero and at infinity. Then the modular function space L_ρ associated to Φ satisfies $\tilde{N}(L_\rho) < 1$.*

Remark 3.1 *It is not difficult to find examples of functions satisfying the conditions in the above corollary. Besides $\Phi(t) = |t|^p$ for $p > 1$ we can obtain some other examples using the following result* [6]*: Φ is uniformly convex if $\limsup_{t\to 0} \frac{\Phi'(at)}{\Phi'(t)} < 1$ and $\limsup_{t\to\infty} \frac{\Phi'(at)}{\Phi'(t)} < 1$ for every $a \in (0,1)$. It is easy to check that $\Phi(t) = t^2 - \log(1+t^2)$ satisfies these conditions.*

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