QUASI-CONTRACTION MAPPINGS IN MODULAR SPACES

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ABSTRACT

As a generalization to Banach contraction principle, Ćirić introduced the concept of quasi-contraction mappings. In this paper, we investigate these kind of mappings in modular function spaces without the Δ_2-condition. In particular, we prove the existence of fixed points and discuss their uniqueness.

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1. INTRODUCTION

Let (M, d) a metric space. A mapping, $T : M \to M$ is said to be quasi-contraction if there exists $k < 1$ such that

$$d(T(x), T(y)) \leq k \max\Big(d(x, y); d(x, T(x)); d(y, T(y)); d(x, T(y)); d(y, T(x))\Big)$$

for any $x, y \in M$. In 1974 Ćirić [1] introduced these mappings and proved an existence fixed point result very similar to the original Banach contraction fixed point theorem. Recently the authors [12] tried to extend his ideas to modular spaces. Though their conclusions are very similar to Ćirić's results proved in metric spaces, they were unable to escape the Δ_2-condition. They also asked whether Ćirić's results may be proved in the modular setting without the very restrictive Δ_2-condition. In this work, we give a proof in the affirmative.

Recall that modular spaces were initiated by Nakano in 1950 [11] in connection with the theory of order spaces and redefined and generalized by Luxemburg [8, 9] and Orlicz in 1959. These spaces were developed following the successful theory of Orlicz spaces, which replaces the particular, integral form of the nonlinear functional, which controls the growth of members of the space, by an abstractly given functional with some good properties. The monographic exposition of the theory of Orlicz spaces may be found in the book of Krasnosel'skii and Rutickii [7] . For a current review of the theory of Musielak-Orlicz spaces and modular spaces the reader is referred to the books of Musielak [10] and Kozlowski [6].

For more on fixed point theory in modular spaces, the reader is advised to consult the references [2],[3], [4], [6], and the references therein.

2. PRELIMINARIES

Let $\mathcal{X}$ be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A functional $\rho : \mathcal{X} \to [0, \infty]$ is called a modular, if for arbitrary f and g, elements of $\mathcal{X}$, there holds :

(1) $\rho(f) = 0$ if and only if $f = 0$;
(2) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$,
(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ whenever $\alpha,\ \beta \geq 0$ and $\alpha + \beta = 1$.

If we replace (3) by

(3') $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ whenever $\alpha,\ \beta \geq 0$ and $\alpha + \beta = 1$.

then the modular ρ is called convex. If ρ is a modular in $\mathcal{X}$ then the set defined by

$$\mathcal{X}_\rho = \{h \in \mathcal{X}\ ; \lim_{\lambda \to 0} \rho(\lambda h) = 0\}$$

is called a modular space. $\mathcal{X}_\rho$ is a vector subspace of $\mathcal{X}$.

Definition 1. *A function modular is said to satisfy the Δ_2-type condition if there exists $K > 0$ such that for any $f \in \mathcal{X}_\rho$ we have $\rho(2f) \leq K\rho(f)$.*

Definition 2. *Let $(\mathcal{X}, \rho)$ be a modular space.*

(1) *The sequence $\{f_n\}_n \subset \mathcal{X}_\rho$ is said to be ρ-convergent to $f \in \mathcal{X}_\rho$ if*

$$\rho(f_n - f) \to 0$$

as $n \to \infty$.

(2) *The sequence $\{f_n\}_n \subset \mathcal{X}_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m go to ∞.*

(3) *A subset C of $\mathcal{X}_\rho$ is called ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belongs to C.*
(4) *A subset C of $\mathcal{X}_\rho$ is called ρ-complete if any ρ-Cauchy sequence in C is ρ-convergent and its ρ-limit is in C.*
(5) *A subset C of $\mathcal{X}_\rho$ is called ρ-bounded if*

$$\delta_\rho(C) = \sup\{\rho(f-g); f, g \in C\} < \infty.$$

The following property is crucial throughout this paper.

Definition 3. *The modular ρ has the Fatou property if and only $\rho(f) \leq \liminf\limits_{n\to\infty} \rho(f_n)$ whenever $\{f_n\}$ ρ-converges to f.*

Note that ρ has the Fatou property if and only if the ρ-ball $B_\rho(f,r) = \{g \in \mathcal{X}_\rho; \rho(f-g) \leq r\}$ is ρ-closed, for any $f \in \mathcal{X}_\rho$ and $r \geq 0$.

3. A Fixed Point Theorem

Similarly to Ćirić definition we introduce the concept of quasi-contractions in modular spaces.

Definition 4. *Let $(\mathcal{X}, \rho)$ be a modular space. Let C be a nonempty subset of $\mathcal{X}_\rho$. The self map $T : C \to C$ is said to be **quasi-contraction** if there exists $k < 1$ such that*

$$\rho(T(x)-T(y)) \leq k \max\Big(\rho(x-y); \rho(x-T(x)); \rho(y-T(y)); \rho(x-T(y)); \rho(y-T(x))\Big)$$

for any $x, y \in C$.

In the sequel we prove an existence fixed point theorem for such mappings. First let T and C as in the above definition. For any $x \in C$ define the orbit

$$\mathcal{O}(x) = \{x, T(x), T^2(x), \cdots\} ,$$

and its ρ-diameter by

$$\delta_\rho(x) = \operatorname{diam}(\mathcal{O}(x)) = \sup\{\rho(T^n(x) - T^m(x)); n, m \in \mathbb{N}\} .$$

Lemma 3.1. *Let $(\mathcal{X}, \rho)$ be a modular space. Let C be a nonempty subset of $\mathcal{X}_\rho$ and $T : C \to C$ be quasi-contraction. Let $x \in C$ such that $\delta_\rho(x) < \infty$. Then for any $n \geq 1$, we have*

$$\delta_\rho(T^n(x)) \leq k^n \delta_\rho(x) ,$$

where k is the constant associated with the quasi-contraction definition of T. Moreover we have

$$\rho(T^n(x) - T^{n+m}(x)) \leq k^n \delta_\rho(x)$$

for any $n \geq 1$ and $m \in \mathbb{N}$.

Proof. Let $n, m \geq 1$, we have

$$\begin{aligned} \rho(T^n(x) - T^m(y)) \quad &\leq k \max \Big(\rho(T^{n-1}(x) - T^{m-1}(y)); \rho(T^{n-1}(x) - T^n(x)); \\ &\rho(T^m(y) - T^{m-1}(y)); \rho(T^{n-1}(x) - T^m(y)); \rho(T^n(x) - T^{m-1}(y)) \Big) \end{aligned}$$

for any $x, y \in C$. This obviously implies the following

$$\delta_\rho(T^n(x)) \leq k\delta_\rho(T^{n-1}(x))$$

for any $n \geq 1$. Hence for any $n \geq 1$, we have

$$\delta_\rho(T^n(x)) \leq k^n \delta_\rho(x) \ .$$

Moreover for any $n \geq 1$ and $m \in \mathbb{N}$, we have

$$\rho(T^n(x) - T^{n+m}(x)) \leq \delta_\rho(T^n(x)) \leq k^n \delta_\rho(x) \ .$$

□

The next lemma will be helpful to prove the main result of this paper.

Lemma 3.2. *Let $(\mathcal{X}, \rho)$ be a modular space such that ρ satisfies the Fatou property. Let C be a ρ-complete nonempty subset of $\mathcal{X}_\rho$ and $T : C \to C$ be quasi-contraction. Let $x \in C$ such that $\delta_\rho(x) < \infty$. Then $\{T^n(x)\}$ ρ-converges to $\omega \in C$. Moreover we have*

$$\rho(T^n(x) - \omega) \leq k^n \delta_\rho(x)$$

for any $n \geq 1$.

Proof. From the previous lemma, we know that $\{T^n(x)\}$ is ρ-Cauchy. Since C is ρ-complete, then there exists $\omega \in C$ such that $\{T^n(x)\}$ ρ-converges to ω. Since

$$\rho(T^n(x) - T^{n+m}(x)) \leq k^n \delta_\rho(x)$$

for any $n \geq 1$, $m \in \mathbb{N}$, and ρ satisfies the Fatou property, we let $m \to \infty$ to get

$$\rho(T^n(x) - \omega) \leq k^n \delta_\rho(x) \ .$$

□

Next we prove that ω is in fact a fixed point of T and it is unique provided some extra assumptions.

Theorem 3.1. *Let C, T, and x be as in the previous Lemma. Assume $\rho(\omega - T(\omega)) < \infty$ and $\rho(x - T(\omega)) < \infty$. Then the ρ-limit ω of $\{T^n(x)\}$ is a fixed point of T, i.e. $T(\omega) = \omega$. Moreover if ω^* is any fixed point of T in C such that $\rho(\omega - \omega^*) < \infty$, then we have $\omega = \omega^*$.*

Proof. We have

$$\rho(T(x)-T(\omega)) \leq k \max\Big(\rho(x-\omega); \rho(x-T(x)); \rho(T(\omega)-\omega); \rho(T(x)-\omega); \rho(x-T(\omega))\Big).$$

From the previous results, we get

$$\rho(T(x) - T(\omega)) \leq k \max\Big(\delta_\rho(x); \rho(\omega - T(\omega)); \rho(x - T(\omega))\Big).$$

Assume that for $n \geq 1$ we have

$$\rho(T^n(x) - T(\omega)) \leq \max\Big(k^n \delta_\rho(x); k\rho(\omega - T(\omega)); k^n \rho(x - T(\omega))\Big).$$

Then

$$\begin{aligned} \rho(T^{n+1}(x) - T(\omega)) \leq \quad & k \max\Big(\rho(T^n(x) - \omega); \rho(T^n(x) - T^{n+1}(x)); \\ & \rho(\omega - T(\omega)); \rho(T^{n+1}(x) - \omega); \rho(T^n(x) - T(\omega))\Big) . \end{aligned}$$

Hence

$$\rho(T^{n+1}(x) - T(\omega)) \leq k \max\Big(k^n \delta_\rho(x); \rho(\omega - T(\omega)); \rho(T^n(x) - T(\omega))\Big) .$$

Using our previous assumption, we get

$$\rho(T^{n+1}(x) - T(\omega)) \leq \max\Big(k^{n+1} \delta_\rho(x); k\rho(\omega - T(\omega)); k^{n+1} \rho(x - T(\omega))\Big) .$$

So by induction, we have

$$\rho(T^n(x) - T(\omega)) \leq \max\Big(k^n \delta_\rho(x); k\rho(\omega - T(\omega)); k^n \rho(x - T(\omega))\Big)$$

for any $n \geq 1$. Therefore we have

$$\limsup_{n\to\infty} \rho(T^n(x) - T(\omega)) \leq k\rho(\omega - T(\omega)) .$$

Using the Fatou property satified by ρ we get

$$\rho(\omega - T(\omega)) \leq \liminf_{n\to\infty} \rho(T^n(x) - T(\omega)) \leq k\rho(\omega - T(\omega)) \ .$$

Since $k < 1$, we get $\rho(\omega - T(\omega)) = 0$ or $T(\omega) = \omega$. Let ω^* be another fixed point of T such that $\rho(\omega - \omega^*) < \infty$. Then we have

$$\rho(\omega - \omega^*) = \rho(T(\omega) - T(\omega^*)) \leq k\rho(\omega - \omega^*)$$

which implies $\rho(\omega - \omega^*) = 0$ or $\omega = \omega^*$. This completes the proof of our theorem. □

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