

ON ASYMPTOTIC POINTWISE CONTRACTIONS IN METRIC SPACES

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Abstract

We discuss the existence of fixed points of asymptotic pointwise mappings in metric spaces. This is the nonlinear version of some known results proved in Banach spaces. We also discuss the case of multivalued mappings.

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1. INTRODUCTION

The notion of asymptotic pointwise mappings was introduced in [13, 14, 15]. The use of ultrapower technique was useful in proving some related fixed point results. In a recent paper [15], the authors gave simple and elementary proofs for the existence of fixed point theorems for asymptotic pointwise mappings without the use of ultrapowers. In this paper, we extend most of their results to metric spaces. In particular we consider the case of CAT(0) as an example of uniform convex metric spaces. In this work, we show how weak-compactness in Banach spaces is extended to metric spaces.

For more on metric fixed point theory, the reader may consult the book [12].

2. BASIC DEFINITIONS AND RESULTS

First let us start by making some basic definitions.

Definition 1. Let (M, d) be a metric space. A mapping $T : M \rightarrow M$ is called a *pointwise contraction* if there exists a mapping $\alpha : M \rightarrow [0, 1)$ such that

$$d(T(x), T(y)) \leq \alpha(x)d(x, y) \text{ for any } y \in M.$$

It is clear that pointwise contractive behavior was introduced to extend the contractive behavior in Banach contraction principle. The central fixed point result for such mappings is the following theorem

Theorem 2.1. ([13, 14]). Let K be a weakly compact convex subset of a Banach space and suppose $T : K \rightarrow K$ is a pointwise contraction. Then T has a unique fixed point, x_0 . Moreover the orbit $\{T^n(x)\}$ converges to x_0 , for each $x \in M$.

Note that if T is a pointwise contraction, then it is continuous. Moreover if $\alpha(x) = 0$ for some $x \in M$, then T is a constant map. Since the main focus of this paper is about the fixed point problem, it is easy to prove that a pointwise contraction $T : M \rightarrow M$ has at most one fixed point, and if a is its fixed point, then the orbit $\{T^n(x)\}$ converges to a , for each $x \in M$. Indeed, we have

$$d(a, T^n(x)) \leq \alpha(a)^n d(a, x)$$

for any $x \in M$. The above conclusion follows because $\alpha(a) < 1$. It is not clear how to prove the existence of the fixed point from the convergence of the orbits which is the case in the classical proof given to the Banach contraction principle.

Definition 2. Let (M, d) be a metric space. A mapping $T : M \rightarrow M$ is called an *asymptotic pointwise mapping* if there exists a sequence of mappings $\alpha_n : M \rightarrow [0, \infty)$ such that

$$d(T^n(x), T^n(y)) \leq \alpha_n(x)d(x, y) \text{ for any } y \in M.$$

- (i) If $\{\alpha_n\}$ converges pointwise to $\alpha : M \rightarrow [0, 1)$, then T is called an *asymptotic pointwise contraction*.
- (ii) If $\limsup_{n \rightarrow \infty} \alpha_n(x) \leq 1$, then T is called *asymptotic pointwise nonexpansive*.
- (ii) If $\limsup_{n \rightarrow \infty} \alpha_n(x) \leq k$, with $0 < k < 1$, then T is called *strongly asymptotic pointwise contraction*.

3. POINTWISE CONTRACTIONS IN METRIC SPACES

Let M be a metric space and \mathcal{F} a family of subsets of M . Then we say that \mathcal{F} defines a *convexity structure* on M if it contains the closed balls and is stable by intersection. For instance $\mathcal{A}(M)$, the class of the admissible subsets of M , defines a convexity structure on any metric space M . Recall that a subset of M is admissible if it is a nonempty intersection of closed balls.

At this point we introduce some notation which will be used throughout the remainder of this work. For a subset A of a metric space M , set:

$$\begin{aligned} r_x(A) &= \sup\{d(x, y) : y \in A\}, \quad x \in M; \\ R(A) &= \inf\{r_x(A) : x \in A\}; \\ \text{diam}(A) &= \sup\{d(x, y) : x, y \in A\}; \\ C_A(A) &= \{x \in A : r_x(A) = R(A)\}; \\ \text{cov}(A) &= \bigcap\{B : B \text{ is a ball and } B \supseteq A\}. \end{aligned}$$

$\text{diam}(A)$ is called the *diameter* of A , $R(A)$ is called the *Chebyshev radius* of A , $C_A(A)$ is called the *Chebyshev center* of A , and $\text{cov}(A)$ is called the *cover* of A .

Definition 3. Let \mathcal{F} be a convexity structure on M .

- (i) We will say that \mathcal{F} is *compact* if any family $(A_\alpha)_{\alpha \in \Gamma}$ of elements of \mathcal{F} , has a nonempty intersection provided $\bigcap_{\alpha \in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$.
- (ii) We will say that \mathcal{F} is *normal* if for any $A \in \mathcal{F}$, not reduced to one point, we have $R(A) < \text{diam}(A)$.
- (iii) We will say that \mathcal{F} is *uniformly normal* if there exists $c \in (0, 1)$ such that for any $A \in \mathcal{F}$, not reduced to one point, we have $R(A) \leq c \text{diam}(A)$. It is easy to check that $c \geq 1/2$.

Example 1. A metric space M is said to be *hyperconvex* [1] if given any family $\{x_\alpha\}$ of points of M and any family $\{r_\alpha\}$ of nonnegative real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

it is the case that $\cap_{\alpha} B(x_{\alpha}; r_{\alpha}) \neq \emptyset$. It is well known [2, 7] that if M is hyperconvex, then $\mathcal{A}(M)$ is compact and uniformly normal with

$$R(A) = \frac{1}{2} \text{diam}(A)$$

for any $A \in \mathcal{A}(M)$.

The main result of [13, 14] may be stated in metric spaces as follows

Theorem 3.1. *Let M be a bounded metric space. Assume that the convexity structure $\mathcal{A}(M)$ is compact. Let $T : M \rightarrow M$ be a pointwise contraction. Then T has a unique fixed point, x_0 . Moreover the orbit $\{T^n(x)\}$ converges to x_0 , for each $x \in M$.*

Proof. Since $\mathcal{A}(M)$ is compact, there exists a minimal nonempty $K \in \mathcal{A}(M)$ such that $T(K) \subset K$. It is easy to check that $\text{cov}(T(K)) = K$. Let $a \in K$, then we have $K \subset B(a, r_a(K))$. Since T is pointwise contraction, there exists a mapping $\alpha : M \rightarrow [0, 1)$ such that

$$d(T(x), T(y)) \leq \alpha(x)d(x, y) \text{ for any } y \in M.$$

In particular, we have then $T(K) \subset B(T(a), \alpha(a)r_a(K))$, which implies $\text{cov}(T(K)) \subset B(T(a), \alpha(a)r_a(K))$. So $r_{T(a)}(K) \leq \alpha(a)r_a(K)$. This will force $\text{diam}(K) = 0$. Indeed let $a \in K$ and define

$$K_a = \{x \in K; r_x(K) \leq r_a(K)\}.$$

Clearly K_a is not empty. Moreover we have

$$K_a = \bigcap_{x \in K} B(x, r_a(K)) \cap K \in \mathcal{A}(M).$$

And since $r_{T(a)}(K) \leq \alpha(a)r_a(K)$, we get $T(K_a) \subset K_a$. The minimality behavior of K implies $K_a = K$. In particular we have $r_x(K) = r_a(K)$ for any $x \in K$. Hence $\text{diam}(K) = r_a(K)$, for any $a \in K$, i.e. a is a diametral point of K . Hence $\text{diam}(K) \leq \alpha(a)\text{diam}(K)$. And since $\alpha(a) < 1$, we get $\text{diam}(K) = 0$, i.e. K is reduced to one point which is fixed by T . Hence the fixed point set of T is not empty. The remaining conclusion of the theorem follows from the general properties of pointwise contractions. \square

It is well known [2, 7] that if M is hyperconvex, then $\mathcal{A}(M)$ is compact, hence we obtain:

Corollary 1. *Let M be a bounded hyperconvex metric space. Let $T : M \rightarrow M$ be a pointwise contraction. Then T has a unique fixed point, x_0 . Moreover the orbit $\{T^n(x)\}$ converges to x_0 , for each $x \in M$.*

4. ASYMPTOTICALLY POINTWISE CONTRACTIONS

Let M be a metric space and \mathcal{F} a convexity structure. We will say that a function $\Phi : M \rightarrow [0, \infty)$ is \mathcal{F} -convex if $\{x; \Phi(x) \leq r\} \in \mathcal{F}$ for any $r \geq 0$. Also we define a type to be a function $\Phi : M \rightarrow [0, \infty)$ defined as

$$\Phi(u) = \limsup_{n \rightarrow \infty} d(x_n, u)$$

where (x_n) is a bounded sequence in M . Types are very useful in the study of the geometry of Banach spaces and the existence of fixed point of mappings. We will say that a convexity structure \mathcal{F} on M is T-stable if types are \mathcal{F} -convex. We have the following lemma.

Lemma 4.1. *Let M be a metric space and \mathcal{F} a compact convexity structure on M which is T-stable. Then for any type Φ , there exists $x_0 \in M$ such that*

$$\Phi(x_0) = \inf\{\Phi(x); x \in M\}.$$

The proof is easy and will be omitted.

Theorem 4.1. *Let M be a bounded metric space. Assume that the convexity structure $\mathcal{A}(M)$ is compact. Let $T : M \rightarrow M$ be a strongly asymptotic pointwise contraction. Then T has a unique fixed point, x_0 . Moreover the orbit $\{T^n(x)\}$ converges to x_0 , for each $x \in M$.*

Proof. First note that T has at most one fixed point. Indeed, let $a, b \in M$ be two fixed points of T . Then we have

$$d(a, b) = d(T^n(a), T^n(b)) \leq \alpha_n(a)d(a, b).$$

If we let n go to infinity, we get $d(a, b) \leq kd(a, b)$ for some $k \in (0, 1)$. This will force $d(a, b) = 0$. Next let $x \in M$ and define the type

$$\Phi(u) = \limsup_{n \rightarrow \infty} d(T^n(x), u), \text{ for each } u \in M.$$

Since $\mathcal{A}(M)$ is compact, then

$$\Omega(x) = \bigcap_{n \geq 1} \text{cov}\left(\{T^k(x); k \geq n\}\right) \neq \emptyset.$$

Let $\omega \in \Omega(x)$. Then we have

$$d(T^{m+n+h}(x), T^{m+h}(x)) \leq \alpha_h(T^m(x))d(T^n(x), T^m(x)).$$

If we let n go to infinity, we get

$$\Phi(T^{m+h}(x)) \leq \alpha_h(T^m(x))\Phi(T^m(x)).$$

Next we let h go to infinity to get

$$\limsup_{n \rightarrow \infty} \Phi(T^n(x)) \leq k\Phi(T^n(x))$$

for some $k \in (0, 1)$, which easily implies that $\limsup_{n \rightarrow \infty} \Phi(T^n(x)) = 0$. Next we notice that

$$\Phi(\omega) \leq \limsup_{n \rightarrow \infty} \Phi(T^n(x)) = 0.$$

Indeed let $u \in M$, then for any $\varepsilon > 0$, then there exists $n_0 \geq 1$ such that for any $n \geq n_0$

$$d(T^n(x), u) \leq \Phi(u) + \varepsilon.$$

In particular we have $T^n(x) \in B(u, \Phi(u) + \varepsilon)$, for any $n \geq n_0$. So

$$\Omega(x) \subset \text{cov}\left(\{T^n(x); n \geq n_0\}\right) \subset B(u, \Phi(u) + \varepsilon),$$

which implies $\omega \in B(u, \Phi(u) + \varepsilon)$. This is true for any $\varepsilon > 0$. Hence for any $u \in M$ we have $d(\omega, u) \leq \Phi(u)$. Hence

$$\Phi(\omega) = \limsup_{n \rightarrow \infty} d(T^n(x), \omega) \leq \limsup_{n \rightarrow \infty} \Phi(T^n(x)).$$

Therefore we have $\Phi(\omega) = 0$ which implies that $\{T^n(x)\}$ converges to ω . This will force ω to be a fixed point of T . Since we already noticed that T has at most one fixed point, then T has a fixed point x_0 and any orbit converges to x_0 . \square

If M is hyperconvex, then $\mathcal{A}(M)$ is compact, hence we obtain:

Corollary 2. *Let M be a bounded hyperconvex metric space. Let $T : M \rightarrow M$ be a strongly asymptotic pointwise contraction. Then T has a unique fixed point, x_0 . Moreover the orbit $\{T^n(x)\}$ converges to x_0 , for each $x \in M$.*

Next we relax the strong behavior of T but assume that types are convex to obtain the following result

Theorem 4.2. *Let M be a bounded metric space. Assume that there exists a convexity structure \mathcal{F} which is compact and T -stable. Let $T : M \rightarrow M$ be an asymptotic pointwise contraction. Then T has a unique fixed point, x_0 . Moreover the orbit $\{T^n(x)\}$ converges to x_0 , for each $x \in M$.*

Proof. Similarly one can easily show that T has at most one fixed point. As we did in the proof of the previous result, let $x \in M$ and define the type

$$\Phi(u) = \limsup_{n \rightarrow \infty} d(T^n(x), u), \text{ for each } u \in M.$$

Since \mathcal{F} is compact and T -stable, then there exists $x_0 \in M$ such that

$$\Phi(x_0) = \inf\{\Phi(u); u \in M\}.$$

Let us show that $\Phi(x_0) = 0$. Indeed we have

$$d(T^{n+m}(x), T^m(x_0)) \leq \alpha_m(x_0)d(T^n(x), x_0),$$

for any $n, m \geq 1$. If we let n go to infinity, we get

$$\Phi(T^m(x_0)) \leq \alpha_m(x_0)\Phi(x_0)$$

which implies

$$\Phi(x_0) = \inf\{\Phi(u); u \in M\} \leq \Phi(T^m(x_0)) \leq \alpha_m(x_0)\Phi(x_0).$$

If we let m go to infinity, we get $\Phi(x_0) \leq \alpha(x_0)\Phi(x_0)$. Since $\alpha(x_0) < 1$, we get $\Phi(x_0) = 0$, which implies that $\{T^n(x)\}$ converges to x_0 . This will force x_0 to be a fixed point of T . Since we already noticed that T has at most one fixed point, then T has a fixed point x_0 and any orbit converges to x_0 . \square

5. ASYMPTOTIC POINTWISE NONEXPANSIVE MAPPINGS

We should note that any result on asymptotic pointwise nonexpansive mappings in metric spaces should extend what is known on asymptotic nonexpansive mappings in metric spaces. Unfortunately only a partial result in hyperconvex

metric spaces is known (see [11]). Here we extend the result found in [15] in uniformly Banach spaces to CAT(0) metric spaces. Indeed these metric spaces offer a nice example of uniformly convex metric spaces. It is not clear that the main inequality used in [15] satisfied in uniformly convex Banach spaces has a similar one in uniformly convex metric spaces.

A metric space (X, d) is said to be a *length space* if each two points of X are joined by a rectifiable path (that is, a path of finite length) and the distance between any two points of X is taken to be the infimum of the lengths of all rectifiable paths joining them. In this case, d is said to be a *length metric* (otherwise known as an *inner metric* or *intrinsic metric*). In case no rectifiable path joins two points of the space the distance between them is said to be ∞ .

A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y . (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic. X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denote by $[x, y]$ called the segment joining x to y .

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in M_κ^2 such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. If $\kappa > 0$ it is further assumed that the perimeter of $\Delta(x_1, x_2, x_3)$ is less than $2D_\kappa$, where D_κ denotes the diameter of M_κ^2 . Such a triangle always exists.

A geodesic metric space is said to be a CAT(κ) space if all geodesic triangles of appropriate size satisfy the following CAT(κ) comparison axiom.

CAT(κ): Let Δ be a geodesic triangle in X and let $\bar{\Delta} \subset M_\kappa^2$ be a comparison triangle for Δ . Then Δ is said to satisfy the CAT(κ) *inequality* if for all $x, y \in \Delta$

and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete CAT(0) spaces are often called *Hadamard spaces*. These spaces are of particular relevance to this study.

Finally we observe that if x, y_1, y_2 are points of a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the CAT(0) inequality implies

$$(1) \quad d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2$$

because equality holds in the Euclidean metric. In fact (see [3], p. 163), *a geodesic metric space is a CAT(0) space if and only if it satisfies inequality (1)* (which is known as the CN inequality of Bruhat and Tits [4]). Moreover if M is a CAT(0) metric space and $x, y \in M$, then for any $\alpha \in [0, 1]$ there exists a unique point $\alpha x \oplus (1 - \alpha)y \in [x, y]$ such that

$$(2) \quad d\left(z, \alpha x \oplus (1 - \alpha)y\right) \leq \alpha d(z, x) + (1 - \alpha)d(z, y), \quad \text{for any } z \in M,$$

and $[x, y] = \{\alpha x \oplus (1 - \alpha)y, \alpha \in [0, 1]\}$.

Let M be a complete CAT(0) space. A subset $C \subset M$ is convex if for any $x, y \in C$ we have $[x, y] \subset C$. Denote by $\mathcal{C}(M)$ the family of all closed convex subsets of M . Then $\mathcal{C}(M)$ defines a convexity structure which is compact and uniformly normal [6]. Note that any type function is convex, i.e. $\mathcal{C}(M)$ is T-convex. This follows easily from the inequality (2). A direct implication of these properties is that any type function achieves its infimum, i.e. for any bounded sequence $\{x_n\}$ in a CAT(0) space M , there exists $\omega \in M$ such that $\Phi(\omega) = \inf\{\Phi(x); x \in M\}$, where

$$\Phi(x) = \limsup_{n \rightarrow \infty} d(x_n, x).$$

Theorem 5.1. *Let M be a complete CAT(0) metric space. Let C be a bounded closed nonempty convex subset of M . Then any $T : C \rightarrow C$ pointwise asymptotically nonexpansive has a fixed point. The fixed point set $Fix(T)$ is closed and convex, i.e. $Fix(T) \in \mathcal{C}(M)$.*

Proof. As before, let $x \in C$ and define the type

$$\Phi(u) = \limsup_{n \rightarrow \infty} d(T^n(x), u), \quad \text{for any } u \in C.$$

Let $\omega \in C$ such that $\Phi(\omega) = \inf\{\Phi(u); u \in C\} = \Phi_0$. We have seen that $\Phi(T^n(\omega)) \leq \alpha_n(\omega)\Phi(\omega) = \alpha_n(\omega)\Phi_0$, for any $n \geq 1$. The CN inequality implies

$$\begin{aligned} d\left(T^n(x), \frac{T^m(\omega) \oplus T^h(\omega)}{2}\right)^2 &\leq \frac{1}{2}d(T^n(x), T^m(\omega))^2 + \frac{1}{2}d(T^n(x), T^h(\omega))^2 \\ &\quad - \frac{1}{4}d(T^m(\omega), T^h(\omega))^2. \end{aligned}$$

If we let n go to infinity, we get

$$\Phi_0^2 \leq \Phi\left(\frac{T^m(\omega) \oplus T^h(\omega)}{2}\right)^2 \leq \frac{1}{2}\Phi(T^m(\omega))^2 + \frac{1}{2}\Phi(T^h(\omega))^2 - \frac{1}{4}d(T^m(\omega), T^h(\omega))^2,$$

which implies

$$d(T^m(\omega), T^h(\omega))^2 \leq \Phi_0^2(2\alpha_m(\omega)^2 + 2\alpha_h^2(\omega) - 4).$$

Since T is pointwise asymptotic nonexpansive, we get

$$\limsup_{m, h \rightarrow \infty} (T^m(\omega), T^h(\omega))^2 \leq 0,$$

which implies $\{T^n(\omega)\}$ is a Cauchy sequence. Let $v = \lim_{n \rightarrow \infty} T^n(\omega)$. Since T is continuous, then $T(v) = v$, i.e. v is a fixed point of T . This proves that $Fix(T)$ is not empty. Again since T is continuous, $Fix(T)$ is closed. In order to prove that $Fix(T)$ is convex, it is enough to prove that $\frac{x \oplus y}{2} \in Fix(T)$, whenever $x, y \in Fix(T)$. Indeed set $\omega = \frac{x \oplus y}{2}$. The CN inequality implies

$$d(T^n(\omega), \omega)^2 \leq \frac{1}{2}d(x, T^n(\omega))^2 + \frac{1}{2}d(y, T^n(\omega))^2 - \frac{1}{4}d(x, y)^2,$$

for any $n \geq 1$. Since

$$d(x, T^n(\omega))^2 = d(T^n(x), T^n(\omega))^2 \leq \alpha_n^2(\omega) d(\omega, x)^2 = \frac{\alpha_n^2(\omega) d(x, y)^2}{4},$$

and

$$d(y, T^n(\omega))^2 = d(T^n(y), T^n(\omega))^2 \leq \alpha_n^2(\omega) d(\omega, y)^2 = \frac{\alpha_n^2(\omega) d(x, y)^2}{4},$$

we get

$$d(T^n(\omega), \omega)^2 \leq \frac{(\alpha_n^2(\omega) - 1) d(x, y)^2}{4},$$

for any $n \geq 1$. Since T is pointwise asymptotic nonexpansive, we get $\lim_{n \rightarrow \infty} T^n(\omega) = \omega$, which implies $T(\omega) = \omega$, i.e. $\omega \in \text{Fix}(T)$. \square

If U, V are bounded subsets of a metric space X , let H denote the Hausdorff metric defined as usual by

$$H(U, V) = \inf\{\varepsilon > 0 : U \subset N_\varepsilon(V) \text{ and } V \subset N_\varepsilon(U)\},$$

where $N_\varepsilon(V) = \{y \in X : d(y, V) < \varepsilon\}$. Let E be a subset of a metric space X . A mapping $T : E \rightarrow 2^X$ with nonempty bounded values is nonexpansive provided $H(T(x), T(y)) \leq d(x, y)$ for all $x, y \in E$. Let $t : E \rightarrow E$ and $T : E \rightarrow 2^X$ with $T(x) \cap E \neq \emptyset$ for $x \in E$. Then t and T are said to be commuting mappings if $t(y) \in T(t(x)) \cap E$ for all $y \in T(x) \cap E$ and for all $x \in E$. A point $z \in X$ is called a center [8] for a mapping $t : E \rightarrow X$ if for each $x \in E$, $d(z, t(x)) \leq d(z, x)$. The set $Z(t)$ denotes the set of all centers of the mapping t .

As an application of our Theorem 5.1 we obtain:

Theorem 5.2. *Let M be a complete CAT(0) space and C be a bounded closed convex subset of M . Assume $t : C \rightarrow C$ is pointwise asymptotically nonexpansive and $T : C \rightarrow 2^C$ is nonexpansive mappings with $T(x)$ a compact convex subset of C for each $x \in C$. If the mappings t and T commute and satisfy the condition*

$$T(x) \cap \text{Fix}(t) \subset Z(t), \text{ for all } x \in \text{Fix}(t)$$

then there is $z \in C$ such that $z = t(z) \in T(z)$.

Proof. By Theorem 5.1, a pointwise asymptotically nonexpansive self-mapping t of a bounded closed convex subset has a nonempty fixed point set A which is a closed convex subset of M . Since t and T commute, $t(y) \in T(t(x)) = T(x)$ for $y \in T(x)$ and $x \in A$, and therefore, $T(x)$ is invariant under t for each $x \in A$. Since $T(x)$ is a closed bounded convex subset of a $CAT(0)$ space, t has a fixed point in $T(x)$ and $T(x) \cap A \neq \emptyset$ for $x \in A$. Now consider the mapping $T(\cdot) \cap A : A \rightarrow$ compact convex subsets of A . We claim that this mapping is nonexpansive. Indeed, if $u \in T(x) \cap A$ for some $x \in A$, let v be the unique closest point in $T(y)$ to u for some $y \in A$. Then $d(u, v) = \inf\{d(u, w); w \in T(y)\}$. However, since $u \in Z(t)$, $d(u, t(v)) \leq d(u, v)$, which contradicts the uniqueness of v as the closest point to u . Therefore, $v = t(v)$ implying $v \in T(y) \cap A$. Since this argument is symmetric in the points x and y , it follows that

$$H(T(x) \cap A, T(y) \cap A) \leq H(T(x), T(y)) \leq d(x, y) \text{ for } x, y \in A.$$

By [10] the nonexpansive mapping $T(\cdot) \cap A : A \rightarrow$ compact convex subsets of A has a fixed point $z \in T(z) \cap A$. Therefore, $z = t(z) \in T(z)$. \square

The Theorem 5.2 may be seen as an extension to Theorem 4.1 of [5].

Corollary 3. *Let M be a complete $CAT(0)$ space and C be a bounded closed convex subset of M . Assume $t : C \rightarrow C$ is pointwise asymptotically nonexpansive and $T : C \rightarrow 2^C$ is nonexpansive mappings with $T(x)$ a compact convex subset of C for each $x \in C$. If the mappings t and T satisfy the condition*

$$\emptyset \neq T(x) \cap \text{Fix}(t) \subset Z(t), \text{ for all } x \in \text{Fix}(t)$$

then there is $z \in C$ such that $z = t(z) \in T(z)$.

A close look at the above proof suggests that a pointwise asymptotic mapping in $CAT(0)$ metric space is may be demi-closed as it was noticed by Göhde [9] for nonexpansive mapping in uniformly Banach spaces. Before we state the next and final result of this work, we need the following notation

$$\{x_n\} \rightharpoonup \omega \text{ if and only if } \Phi(\omega) = \inf_{x \in C} \Phi(x),$$

where C is a closed convex subset which contains the bounded sequence $\{x_n\}$ and $\Phi(x) = \limsup_{n \rightarrow \infty} d(x_n, x)$.

Proposition 1. *Let M be a CAT(0) metric space. Let C be a bounded closed nonempty convex subset of M . Let $T : C \rightarrow C$ be a pointwise asymptotic nonexpansive mapping. Let $\{x_n\} \in C$ be an approximate fixed point sequence, i.e. $\lim_{n \rightarrow \infty} d(x_n, T(x_n)) = 0$, and $\{x_n\} \rightarrow \omega$. Then we have $T(\omega) = \omega$.*

Proof. Since $\{x_n\}$ is an approximate fixed point sequence, then we have

$$\Phi(x) = \limsup_{n \rightarrow \infty} d(T^m(x_n), x)$$

for any $m \geq 1$. Hence $\Phi(T^m(x)) \leq \alpha_m(x)\Phi(x)$, for each $x \in C$. In particular, we have $\lim_{m \rightarrow \infty} \Phi(T^m(\omega)) = \Phi(\omega)$. The CN inequality implies

$$d\left(x_n, \frac{\omega \oplus T^m(\omega)}{2}\right)^2 \leq \frac{1}{2}d(x_n, \omega)^2 + \frac{1}{2}d(x_n, T^m(\omega))^2 - \frac{1}{4}d(\omega, T^m(\omega))^2,$$

for any $n, m \geq 1$. If we let $n \rightarrow \infty$, we will get

$$\Phi\left(\frac{\omega \oplus T^m(\omega)}{2}\right)^2 \leq \frac{1}{2}\Phi(\omega)^2 + \frac{1}{2}\Phi(T^m(\omega))^2 - \frac{1}{4}d(\omega, T^m(\omega))^2,$$

for any $m \geq 1$. The definition of ω implies

$$\Phi(\omega)^2 \leq \frac{1}{2}\Phi(\omega)^2 + \frac{1}{2}\Phi(T^m(\omega))^2 - \frac{1}{4}d(\omega, T^m(\omega))^2,$$

for any $m \geq 1$, or

$$d(\omega, T^m(\omega))^2 \leq 2\Phi(T^m(\omega))^2 - 2\Phi(\omega)^2.$$

If we let $m \rightarrow \infty$, we will get $\lim_{m \rightarrow \infty} d(\omega, T^m(\omega)) = 0$. Hence $T(\omega) = \omega$ since T is continuous. \square

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