SELECTION THEOREMS FOR MULTIFUNCTIONS IN
PARTIALLY ORDERED SETS

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Abstract. It is shown that an order preserving multivalued mapping $T^*$ of
a complete lattice $X$ which takes values in the space of nonempty externally
complete subsets of $X$ always has an order preserving selection. Several fixed
point results are also obtained.

1. Introduction

This paper focuses on external complete lattice structure, a new concept that
was initially introduced in metric spaces as externally hyperconvex sets by Aron-
szajn and Panitchpakdi in their fundamental paper [1] on hyperconvexity. This
idea developed from the original work of A. Quilliot [5] who introduced the con-
cept of generalized metric structures to show that metric hyperconvexity is in fact
similar to complete lattice structure for ordered sets. In this fashion, Tarski’s
fixed point theorem becomes Sine and Soardi’s fixed point theorem for hyper-
convex metric spaces. For more on this, the reader may consult the references
[3, 4].

Our main result yields the fact that an order preserving set-valued mapping
of a complete lattice set, taking externally complete lattice values, always has
a single valued selection which is order preserving. This is used to prove some
fixed point theorems.

We begin by describing the relevant notation and terminology. Let $(X, \prec)$
be a partially ordered set and $M \subset X$ a non-empty subset. Recall that an
upper (lower) bound for $M$ is an element $p \in X$ with $m \prec p$ ($p \prec m$) for each
$m \in M$; the least-upper (greatest-lower) bound of $M$ will be denoted sup $M$
(resp. $\inf M$). Of course there is, in general, no reason for $\inf M$ and $\sup M$

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to exist. $(X, \prec)$ will be called Dedekind-complete if for every subset $M \subset P$, sup $M$ exists provided $M$ is bounded above. Note that in a Dedekind-complete partial order, any nonempty subset $M$ bounded below has inf $M$. Throughout this work we will omit to mention Dedekind and only use complete instead. This assumption is crucial in proving fixed point results. In this work, we will not assume such assumption. Recall that $M \subset X$ is said to be totally ordered if for every $m_1, m_2 \in M$ we have $m_1 \prec m_2$ or $m_2 \prec m_1$. A totally ordered subset of $X$ is called a chain. For any $m \in X$ define

$$(\leftarrow, m) = \{x \in X; x \prec m\}, \text{ and } [m, \rightarrow) = \{x \in X; m \prec x\}.$$  

Through out this work, we will write $B(m) = (\leftarrow, m)$ or $B(m) = [m, \rightarrow)$.

Recall that a partially ordered set $X$ is called a Tree if for every $m \in X$, the subset $(\leftarrow, m)$ is well ordered with the induced order $\prec$. A subset $Y$ of $X$ is called convex if the segment $[x, y] = \{z \in X; x \prec z \prec y\} \subset Y$ provided $x, y \in Y$. Note that the segment may be empty. A map $F : P \to P$ is order preserving (also called monotone, isotone, or increasing) if $F(x) \prec F(y)$ whenever $x \prec y$.

2. Externally Complete sets

Externally complete subsets of a partially ordered set is a similar concept to the externally hyperconvex subsets introduced by Aronszajn and Panitchpakdi [1].

Definition 1. $(X, \prec)$ be a partially ordered set. A subset $M$ of $X$ is called externally complete if and only if for any family of points $(x_i)_{i \in I}$ such that $I(x_i) \cap I(x_j) \neq \emptyset$ for any $i, j \in I$, and $I(x_i) \cap M \neq \emptyset$, then

$$\left( \bigcap_{i \in I} I(x_i) \right) \cap M \neq \emptyset,$$

where $I(x) = (\leftarrow, x)$ or $I(x) = [x, \rightarrow)$.

Note that if $\cap I(x_i) \neq \emptyset$, then we do not need to assume that they do intersect 2-by-2, i.e. $I(x_i) \cap I(x_j) \neq \emptyset$ for any $i, j \in I$. The later condition becomes obvious. The family of all nonempty externally complete subsets of $X$ will be denoted by $EC(X)$. 
Proposition 1. Let $X$ be a partial order. Then any $M \in EC(X)$ is complete and convex.

Proof. Let $A \subset M$ be nonempty and bounded above. Set $U(A) = \{b \in X; A \subset (\leftarrow, b]\}$. Since $A$ is bounded above, the subset $U(A)$ is not empty. It is clear that the families $(I(a))_{a \in A}$, where $I(a) = [a, \rightarrow)$ and $(I(b))_{b \in U(A)}$, where $I(b) = (\leftarrow, b]$ intersect 2-by-2. Moreover we have $I(a) \cap M \neq \emptyset$ and $I(b) \cap M \neq \emptyset$, for any $(a, b) \in A \times U(a)$. Since $M$ is in $EC(X)$, we conclude that 

$$\left( \bigcap_{(a, b) \in A \times U(A)} I(a) \cap I(b) \right) \cap M \neq \emptyset.$$ 

Let $m \in M$ belong to the above intersection. Then for any $a \in A$, we have $a \prec m$. So $m$ is an upper bound of $A$. Let $b$ be any upper bound of $A$, then $b \in U(A)$. Hence $m < b$ or $m$ is the lowest upper bound of $A$, i.e. $m = \sup A$.

Similarly one can prove that $\inf A \in M$ also exists provided $A$ is bounded below. Next we prove that $M$ is convex. Let $x, y \in M$. Obviously if $x$ and $y$ are not comparable, then $[x, y] = \emptyset$ and we have nothing to prove. So assume $x < y$. let $a \in [x, y]$. Obviously we have $(\leftarrow, a] \cap [a, \rightarrow) = \{a\}$. And since $(\leftarrow, a] \cap M \neq \emptyset$ and $[a, \rightarrow) \cap M \neq \emptyset$, then 

$$(\leftarrow, a] \cap [a, \rightarrow) \cap M = \{a\} \cap M \neq \emptyset.$$ 

This obviously implies that $a \in M$, i.e. $[x, y] \subset M$, which completes the proof of our Proposition. \qed

Example 1. Let $\mathbb{N} = \{0, 1, \cdots\}$. we consider the order $0 < 2 < 4 < \cdots$ and $0 < 1 < 3 < \cdots$, and no even number (different from 0) is comparable to any odd number. Then $(\mathbb{N}, \prec)$ is a Tree. Set $M = \{0, 1, 2\}$ is in $EC(\mathbb{N})$. Note that $M$ is not linearly ordered though it is convex.

In the next result we characterize the externally complete subsets of ordered Trees.

Theorem 2.1. Let $X$ be an ordered Tree. A subset $M$ of $X$ is externally complete if and only if $M$ is convex, complete, and any chain has a least upper bound in $M$. 


Proof. Let $M \in \mathcal{EC}(X)$. Then $M$ is convex and complete. Let $C$ be a nonempty chain of $M$. Let $c_1, c_2 \in C$, then we have $c_1 < c_2$ or $c_2 < c_1$. Hence

$$[c_1, \rightarrow) \cap [c_2, \rightarrow) \cap M \neq \emptyset.$$ 

Since $M \in \mathcal{EC}(X)$, then

$$J = \left( \bigcap_{c \in C} [c, \rightarrow) \right) \cap M \neq \emptyset.$$ 

Obviously any $c \in J$ is an upper bound of $M$. Since $M$ is complete, then $\sup C$ exists in $M$. Assume conversely that $M$ is a convex, and complete subset of $X$ such that any chain in $M$ has an upper bound in $M$. Let $x, y \in X$ such that there exist $m_1, m_2 \in M$ with $x < m_1$ and $m_2 < y$. Define $P(x) = \inf \{m \in M; x < m\}$, and $P(y) = \sup \{m \in M; m < y\}$. Both $P(x)$ and $P(y)$ exist and belong to $M$ since $M$ is complete. Let $(x_i)_{i \in I}$ in $X$ such that for any $i \in I$ there exists $m_i \in M$ such that $x_i < m_i$. Also we have $[x_i, \rightarrow) \cap [x_j, \rightarrow) \neq \emptyset$, for any $i, j \in I$. This condition forces the set $\{x_i; i \in I\}$ to be linearly ordered since $X$ is a Tree. Consider the subset $M_I = \{P(x_i); i \in I\}$ of $M$. It is easy to check that $M_I$ is linearly ordered. Since any linearly ordered subset of $M$ is bounded, there exists $m \in M$ such that $P(x_i) < m$ for any $i \in I$. Since $x_i < P(x_i)$ then

$$m \in \left( \bigcap_{i \in I} [x_i, \rightarrow) \right) \cap M \neq \emptyset.$$ 

Next let $(y_j)_{j \in J}$ in $X$ such that for any $j \in J$ there exists $m_j \in M$ such that $m_j < y_j$. Consider the subset $M_J = \{P(y_j); j \in J\}$ of $M$. Since $X$ is Tree, the set $M_J$ is bounded below, so $m_0 = \inf M_J$ exists in $M$. It is obvious that $m_0 < P(y_j) < y_j$ for any $j \in J$. This implies

$$m_0 \in \left( \bigcap_{j \in J} (\leftarrow, y_j]\right) \cap M \neq \emptyset.$$ 

Finally assume that we have $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ in $X$ such that the subsets $\left( [x_i, \rightarrow) \right)_{i \in I}$, and $\left( (\leftarrow, y_j]\right)_{j \in J}$ intersect 2-by-2 and $[x_i, \rightarrow) \cap M \neq \emptyset$ and $(\leftarrow, y_j] \cap M \neq \emptyset$ for any $(i, j) \in I \times J$. As before, set

$$m_I = \sup \{P(x_i); i \in I\} \text{ and } m_J = \inf \{P(y_j); j \in J\}.$$
Since \( x_i < y_j \) for any \((i, j) \in I \times J\), we get \( m_I < m_J \). Obviously we have

\[
[m_I, m_J] \subset \left( \bigcap_{i \in I} [x_i, \to) \right) \bigcap \left( \bigcap_{j \in J} (\leftarrow, y_j) \right) \bigcap M \neq \emptyset.
\]

Hence \( M \) is in \( \mathcal{EC}(X) \). \( \square \)

3. Some Selection Theorems

Before we state our first main result, we need the following definition.

**Definition 2.** Let \( X \) be partial order. Let \( T^* : X \to \mathcal{EC}(X) \) be a multivalued map. For any \( x, y \in X \), Define \( T^*(x) \prec T^*(y) \) whenever \( a \in T^*(x) \), there exists \( b \in T^*(y) \) such that \( a \prec b \) and also for any \( c \in T^*(y) \) there exists \( d \in T^*(x) \) such that \( d \prec c \). Also a point \( x \in X \) is called a fixed point of \( T^* \) if and only if \( x \in T^*(x) \).

Before we proceed to the main result of this work, we will need the concept of weak lattice. Indeed a partial order \( X \) is said to be weak lattice if for any \( x, y \in X \), we have \([x, \to) \cap [y, \to) \neq \emptyset\) and \((\leftarrow, x] \cap (\leftarrow, y] \neq \emptyset\). It is easy to check that if \( M \in \mathcal{EC}(X) \) and \( X \) is weakly lattice, then \( M \) is itself weakly lattice.

**Theorem 3.1.** Let \( X \) be a weakly lattice partial order. Let \( T^* : X \to \mathcal{EC}(X) \) be a multivalued map. Then there exists a map \( T : X \to X \) such that

1. for any \( x \in X \), we have \( T(x) \in T^*(x) \);
2. if \( T^*(x) \prec T^*(y) \) then \( T(x) \prec T(y) \) for any \( x, y \in X \).

Such map \( T \) is called a selection of \( T^* \).

**Proof.** Consider the family \( \mathcal{F} \) of all pairs \((D, T)\), where \( T : D \to X \), such that \( T(d) \in T^*(d) \) for all \( d \in D \), and \( T(x) \prec T(y) \) whenever \( T^*(x) \prec T^*(y) \) for each \( x, y \in D \).

Notice that \( \mathcal{F} \neq \emptyset \) since \((\{x_0\}, T) \in \mathcal{F} \) for any choice of \( x_0 \in X \) and \( T(x_0) \in T^*(x_0) \). Define an order relation on \( \mathcal{F} \) by setting

\[
(D_1, T_1) \preceq (D_2, T_2) \iff D_1 \subset D_2 \text{ and } T_2 \mid_{D_1} = T_1,
\]

where the notation \( T_2 \mid_{D_1} \) means the restriction of \( T_2 \) to \( D_1 \). Let \( \{(D_\alpha, T_\alpha)\} \) be an increasing chain in \( (\mathcal{F}, \preceq) \). Then it follows that \((\bigcup_\alpha D_\alpha, T) \in \mathcal{F} \) where \( T \mid_{D_\alpha} = T_\alpha \).

It is clear that \((\bigcup_\alpha D_\alpha, T)\) is an upper bound of \( \{(D_\alpha, T_\alpha)\} \). By Zorn’s Lemma,
has a maximal element, say \((D, T)\). Assume \(D \neq H\) and select \(x_0 \in X \setminus D\). Set \(\tilde{D} = D \cup \{x_0\}\). Set

\[D_l = \{y \in D; \; T^*(y) \prec T^*(x_0)\}\]

and

\[D_r = \{y \in D; \; T^*(x_0) \prec T^*(y)\}\].

Obviously we have \(T^*(y) \prec T^*(z)\) for any \(y \in D_l\) and \(z \in D_r\). Also since \(X\) is weakly lattice, then \(T^*(x_0) \in EC(X)\) is also a weakly lattice. This will force \(B(T(a)) \cap B(T(b)) \neq \emptyset\) for any \(a, b \in D_l \cup D_r\), where \(B(T(a)) = [T(a), \rightarrow]\) and \(B(T(b)) = (\leftarrow, T(b)]\) for any \(a \in D_l\) and \(b \in D_r\). By definition of \(D_l\) and \(D_r\), we have \(B(T(a)) \cap T^*(x_0) \neq \emptyset\) and \(T^*(x_0) \cap B(T(b)) \neq \emptyset\), for any \((a, b) \in D_l \times D_r\).

Since \(T^*(x_0) \in EC(X)\), then

\[J = \left( \bigcap_{a \in D_l} [T(a), \rightarrow) \right) \cap \left( \bigcap_{b \in D_r} (\leftarrow, T(b)] \right) \cap T^*(x_0) \neq \emptyset.\]

Choose \(y_0 \in J\) and define

\[\tilde{T}(x) = \begin{cases} 
y_0 & \text{if } x = x_0; \\
T(x) & \text{if } x \in D.
\end{cases}\]

Obviously we have \(\tilde{T}(x) \in T^*(x)\) for any \(x \in D \cup \{x_0\}\). Let \(y, z \in D \cup \{x_0\}\) such that \(T^*(y) \prec T^*(z)\). Without loss of any generality assume first \(y = x_0\) and \(z \in D\). Then \(z \in D_r\). From our above construction, we know that \(y_0 = \tilde{T}(x_0) \in (\leftarrow, T(z)]\), i.e. \(\tilde{T}(x_0) \prec \tilde{T}(z)\). Similarly one will show that if \(z = x_0\), then we have \(\tilde{T}(y) \prec \tilde{T}(x_0)\). Therefore \((D \cup \{x_0\}, \tilde{T}) \in \mathcal{F}\) contradicting the maximality of \((D, T)\). As a conclusion we get \(D = X\), which completes the proof of Theorem 3.1.

As a direct consequence we get the following result.

**Theorem 3.2.** Let \(X\) be a weakly lattice partial order. Let \(T^* : X \to EC(X)\) be a multivalued map. Assume that \(T^*\) is order preserving, i.e. \(T^*(x) \prec T^*(y)\) whenever \(x < y\) for any \(x, y \in X\). Then there exists a selection map \(T : X \to X\) which is order preserving.
In the case of order Trees, the weakly lattice assumption forces the Tree to be a totally linearly ordered set. Next we discuss a selection result for Trees without the weak lattice assumption.

**Theorem 3.3.** Let \( X \) be a Tree. Assume that any chain in \( X \) has a supremum. Let \( T^* : X \rightarrow \mathcal{EC}(X) \) be an order preserving multivalued map. Then there exists an order preserving selection map \( T : X \rightarrow X \) of \( T^* \).

**Proof.** Denote by \( e \) the smallest element of \( X \). Consider the family \( \mathcal{F} \) of all pairs \((D,T)\), where \( T : D \rightarrow X \) is an order preserving map, such that \( T(d) \in T^*(d) \) for all \( d \in D \), and \([e,d] \subset D \) for any \( d \in D \). Notice that \( \mathcal{F} \neq \emptyset \) since \((\{e\}, T) \in \mathcal{F} \) for any choice of \( T(e) \in T^*(e) \). Define an order relation on \( \mathcal{F} \) by setting

\[
(D_1, T_1) \preceq (D_2, T_2) \iff D_1 \subset D_2 \text{ and } T_2|_{D_1} = T_1,
\]

where \( T_2|_{D_1} \) is the restriction of \( T_2 \) to \( D_1 \). Let \( \{(D_\alpha, T_\alpha)\} \) be an increasing chain in \((\mathcal{F}, \preceq)\). Let \( d \in \cup_\alpha D_\alpha \), then \( d \in D_{\alpha_0} \) for some \( \alpha_0 \). So \([e, d] \subset \cup_\alpha D_\alpha \) which implies \([e, d] \subset \cup_\alpha D_\alpha \). Hence it follows that \((\cup_\alpha D_\alpha, T) \in \mathcal{F} \) where \( T|_{D_\alpha} = T_\alpha \).

By Zorn’s Lemma, \((\mathcal{F}, \preceq)\) has a maximal element, say \((D, T)\). Assume \( D \neq H \) and select \( x_0 \in X \setminus D \). Since \( X \) is a Tree, set \( x_1 = \min \{x \in [e, x_0]; x \notin D\} \). Then for any \( x \in [e, x_1] \) we have either \( x = x_1 \) or \( x \in D \). Since \( T^* \) is order preserving, then for any \( d \in [e, x_1] \cap D \), there exists \( m \in T^*(x_1) \) such that \( T(d) < m \), i.e \([T(d), \rightarrow) \cap T^*(x_1) \neq \emptyset \). Since \( T \) is order preserving, then for any \( d_1, d_2 \in [e, x_1] \cap D \), \([T(d_1), \rightarrow) \cap [T(d_2), \rightarrow) \neq \emptyset \). Hence

\[
J = \left( \bigcap_{d \in [e, x_1] \cap D} [T(d), \rightarrow) \right) \cap T^*(x_1) \neq \emptyset
\]

since \( T^*(x_1) \) is in \( \mathcal{EC}(X) \). Choose \( y_0 \in J \) and define

\[
\tilde{T}(x) = \begin{cases} 
y_0 & \text{if } x = x_1; \\
T(x) & \text{if } x \in D.
\end{cases}
\]

Obviously we have \((D \cup \{x_1\}, \tilde{T}) \in \mathcal{F} \) contradicting the maximality of \((D, T)\). As a conclusion we get \( D = X \), which completes the proof of Theorem 3.3. \( \square \)

**Remark 1.** In the proof of Theorem 3.3, we used a weaker order on subsets of \( X \). Indeed the conclusion of Theorem 3.3 is still valid if we consider the order
$T^*(x) \preceq e T^*(z)$ if and only if for any $y \in T^*(x)$, there exists $w \in T^*(z)$ such that $y \prec w$. For more on this order please refer to [?].

In the following example, we discuss whether the externally completeness is necessary.

**Example 2.** Let $X = \mathbb{N} \cup \{\omega_n; n \in \mathbb{N}\} \cup \{\omega_\infty\}$ ordered as

$$0 \prec 1 \prec \cdots \prec n \prec \cdots \prec \omega_\infty$$

and $n \prec \omega_n$ and $\omega_n$ is not comparable to any $\omega_i$ and to any $k$ for $n < k$. Obviously $X$ is a Tree. Define the multivalued map $T^*$ by

$$T^*(\omega_\infty) = \{\omega_n; 0 \leq n < \infty\}$$

and $T^*(x) = \{x\}$ for any $x \neq \omega_\infty$. It is easy to check that $T^*$ is an order preserving multivalued map with no order preserving selection.

### 4. Some Fixed Point Theorems

Recall that a partial ordered set $X$ is called a complete lattice if and only if for any nonempty subset $M \subseteq X$, $\sup(M)$ and $\inf(M)$ exist in $X$. the following result is an analogous result to Tarski’s fixed point theorem. We will call it a multivalued Tarski’s fixed point theorem.

**Theorem 4.1.** Let $X$ be a complete lattice. Let $T^*: X \rightarrow \mathcal{EC}(X)$ be an order preserving multivalued map. Then $T^*$ has a fixed point.

**Proof.** Theorem 3.2 implies the existence of an order preserving selection $T$ of $T^*$. Obviously any fixed point of $T$ is also a fixed point of $T^*$. So the classical Tarski’s fixed point theorem will imply that $T^*$ has a fixed point. \(\square\)

As for Trees, we have the following multivalued fixed point theorem.

**Theorem 4.2.** Let $X$ be a Tree. Assume that any chain in $X$ has a supremum. Let $T^*: X \rightarrow \mathcal{EC}(X)$ be an order preserving multivalued map. Then $T^*$ has a fixed point.

**Proof.** First we know from Theorem 3.3 that $T^*$ has an order preserving selection $T$. Let $e$ be the minimum of $X$. Then the transfinite orbit of $e$ will lead to the smallest fixed point of $T$. The existence of this transfinite orbit is assured because any chain in $X$ has a supremum. \(\square\)
For a reference on the above ideas involved originally in Tarski’s fixed point theorem, please refer to [2] and the references therein. As for fixed points of multivalued maps defined on a Tree, the reader is advised to consult [7].

As for the single valued mappings, we may want to find out more about the structure of the fixed point set of a multivalued map. In the following result we only assume that $X$ is a complete partial order, i.e. for any nonempty subset $A \subseteq X$, $\sup(A)$ exists if $A$ is bounded above, and $\inf(A)$ exists if $A$ is bounded below.

**Theorem 4.3.** Let $X$ be a weakly lattice, and let $T^*: X \rightarrow EC(X)$ be an order preserving multivalued map such that the fixed point set $\text{Fix}(T^*) \neq \emptyset$. Then there exists an order preserving selection $T: X \rightarrow X$ of $T^*$ such that $\text{Fix}(T) = \text{Fix}(T^*)$.

**Proof.** Let $\mathfrak{F}$ denote the collection of all pairs $(D, T)$, where $\emptyset \neq D \subseteq \text{Fix}(T^*)$, $T: D \rightarrow X$, $T(d) \in T^*(d)$ for all $d \in D$, $T(x) = x$ for all $x \in \text{Fix}(T^*)$, and $T(x) \preceq T(y)$ whenever $x \preceq y$ for all $x, y \in D$. By assumption $(\text{Fix}(T^*), \text{Id}) \in \mathfrak{F}$, so $\mathfrak{F} \neq \emptyset$. The argument is now a simple modification of the proof of Theorem 3.1. Define an order relation on $\mathfrak{F}$ by setting

$$(D_1, T_1) \preceq (D_2, T_2) \iff D_1 \subset D_2 \text{ and } T_2 |_{D_1} = T_1.$$ 

Let $\{(D_\alpha, T_\alpha)\}$ be an increasing chain in $(\mathfrak{F}, \preceq)$. Then it follows that $(\bigcup_\alpha D_\alpha, T) \in \mathfrak{F}$ where $T |_{D_\alpha} = T_\alpha$. By Zorn’s Lemma, $(\mathfrak{F}, \preceq)$ has a maximal element, say $(D, T)$. Assume $D \neq X$ and select $x_0 \in X \setminus D$. Set $\tilde{D} = D \cup \{x_0\}$ and consider the sets

$$D_l = \{y \in D; \ T^*(y) \prec T^*(x_0)\};$$

and

$$D_r = \{y \in D; \ T^*(x_0) \prec T^*(y)\}.$$ 

As for the proof of Theorem 3.1, one can easily show that

$$J = \left( \bigcap_{a \in D_l} [T(a), \rightarrow) \right) \bigcap \left( \bigcap_{b \in D_r} (\leftarrow, T(b)] \right) \bigcap T^*(x_0) \neq \emptyset.$$ 

Choose $y_0 \in J$ and define

$$\tilde{T}(x) = \begin{cases} y_0 & \text{if } x = x_0; \\ T(x) & \text{if } x \in D. \end{cases}$$
Obviously we have $\tilde{T}(x) \in T^*(x)$ for any $x \in D \cup \{x_0\}$. It is easy to show that $\tilde{T}$ is order preserving. Therefore $(D \cup \{x_0\}, \tilde{T}) \in \mathcal{F}$ contradicting the maximality of $(D, T)$. Therefore $D = X$. So the existence of the order preserving selection $T$ of $T^*$ is assured such that $Fix(T^*) \subset Fix(T)$. But we have $Fix(T) \subset Fix(T^*)$ since $T$ is a selection of $T^*$.

\[ \square \]

**Remark 2.** If $X$ is a complete lattice, then the fixed point set of $T^*$, which satisfies all the assumptions of Theorem 4.1 is a complete lattice. Also one can find many extension to Tarski’s fixed point theorem for mappings which are not order preserving but assume weaker assumptions. We leave it to the readers to prove such results extend easily to the multivalued setting described above.

**References**


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