ON ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN HYPERCONVEX METRIC SPACES

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Abstract. Since bounded hyperconvex metric spaces have the fixed point property for nonexpansive mappings, it is natural to extend such powerful result to asymptotically nonexpansive mappings. Our main result states that the approximate fixed point property holds in this case. The proof is based on the use, for the first time, of the ultrapower of a metric space.

Introduction

The notion of hyperconvexity is due to Aronszajn and Panitchpakdi [AP] who proved that a hyperconvex space is a nonexpansive absolute retract, i.e. it is a nonexpansive retract of any metric space in which it is isometrically embedded. The corresponding linear theory is well developed and associated with the names of Gleason, Goodner, Kelley and Nachbin (see for instance [La]). The nonlinear theory is still developing. The recent interest into these spaces goes back to the results of Sine [Sn1] and Soardi [So] who proved independently that fixed point property for nonexpansive mappings holds in bounded hyperconvex spaces. Since then many interesting results have been shown to hold in hyperconvex spaces. For more on the metric fixed point property, the interested reader may consult [AK] and [GK] as well as the most recent book [KK].

Recall also that Jawhari, Misane and Pouzet [JMP] were able to show that Sine and Soardi's fixed point theorem is equivalent to the classical Tarski's fixed point theorem in complete ordered sets. This happens via the notion of generalized metric spaces. Therefore, the notion of hyperconvexity should be understood and appreciated in a more abstract formulation.

In opposition to the lack of linearity hyperconvexity provides us with a really rich metric structure that leads to a collection of surprising and beautiful results related to different branches of mathematics as, for instance, topology, graph theory, multivalued analysis, fixed point theory.

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In this work, we investigate some open questions related to the fixed point property (fpp) in hyperconvex metric spaces. Historically nonexpansive mappings have enjoyed most of the interest and were at the core of the fpp in hyperconvex metric spaces. The main motivation of this work was a question by Kirk [Ki2] whether asymptotically nonexpansive mappings have the fpp in bounded hyperconvex metric spaces. This question is still open. But we were able to show that asymptotic fixed point property holds in this case. The proof is nonstandard in nature and uses the notion of ultrapower of a metric space. To the best of our knowledge this is the first time that such notion is considered in the metric setting which lead to some positive new results.

Basic Definitions

A metric space M is said to be hyperconvex if given any family $\{x_\alpha\}$ of points of M and any family $\{r_\alpha\}$ of real numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

it is the case that $\cap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$.

The fundamental result of [AP] asserts that a metric space M is hyperconvex if and only if it is *injective*. Thus M is hyperconvex if given any two metric spaces X and Y with Y a subspace of X, and any nonexpansive mapping $f : Y \to M$, then f has a nonexpansive extension $\tilde{f} : X \to M$. Basic results about injective metric spaces can be found in [Is].

An *admissible* subset of M is a set of the form

$$\bigcap_i B(x_i; r_i)$$

where $\{B(x_i; r_i)\}$ is a family of closed balls centered at points $x_i \in M$ with respective radii r_i. It is quite easy to see that an admissible subset of a hyperconvex metric space is hyperconvex. In what follows we use $\mathcal{A}(M)$ to denote the family of all nonempty admissible subsets of M.

The recent interest into hyperconvexity goes back to the results of Sine [Sn1] and Soardi [So] who proved that if H is a bounded hyperconvex metric space and $T : H \to H$ is nonexpansive, i.e. $d(T(x), T(y)) \leq d(x, y)$ for any $x, y \in H$, then there exists a fixed point $x \in H$, i.e. $T(x) = x$. Moreover the fixed point set $Fix(T)$ is hyperconvex and consequently is a nonexpansive retract of H.

May be the most elegant result in this direction goes to Baillon [Ba] who proved that the conclusion of Sine and Soardi results is still valid when dealing with any family of commutative nonexpansive mappings. In fact his proof is based on the following structural result:

Theorem.. *Let H be a bounded hyperconvex metric space. If $\{H_i\}_{i\in I}$ is a decreasing family of nonempty hyperconvex subsets of H, then we have*

$$\bigcap_{i\in I} H_i \neq \emptyset$$

and is hyperconvex.

The proof is non-intuitive and very complicated.

When H is not bounded, then a nonexpansive mappings may not have a fixed point. But it is not hard to see that the nonexpansive mapping T always has an approximate fixed point, i.e.

$$\inf\{d(x,T(x));\ x\in H\} = 0\ .$$

When a map satisfies the above, we say T satisfies the *approximate fixed point property.* In this case, the set

$$H_\varepsilon = \{x\in H;\ d(x,T(x))\leq \varepsilon\}$$

is not empty for any $\varepsilon > 0$. In fact Sine [Sn2] proved that H_ε is hyperconvex.

Next we discuss convexity in hyperconvex metric spaces. Historically there are two approaches to this. One is based on Penot's ideas [Pe] based on the notion of convexity structures who gave the first interesting generalization of the classical Kirk's fixed point theorem [Ki1] in metric spaces. The other one mimics the linear convexity. Here we will use that one. In order to better understand it, we will use a natural embedding of any metric space M into the Banach space $l_\infty(M)$ (see [EK] for more on this). So if H is hyperconvex, then there exists a nonexpansive retract $R : l_\infty(H) \to H$. For any $x,y\in H$, we write

$$tx\oplus(1-t)y = R(tx+(1-t)y)$$

for ant $t\in[0,1]$. Here we are using the linear convexity of $l_\infty(H)$. It is not hard to check that for any $z\in H$ we have

$$d(z, tx\oplus(1-t)y) \leq td(z,x) + (1-t)d(z,y)$$

for ant $t\in[0,1]$.

Ultrapower of Metric Spaces

Let (M,d) be a bounded metric space and $\mathcal{U}$ be a nontrivial ultrafilter on the natural numbers. Consider the cartesian product $\mathcal{M} = \prod\limits_{n\geq 1} M$. Define the equivalence relation $\sim$ on $\mathcal{M}$ by

$$(x_n)\sim(y_n)\ \text{ if and only if }\ \lim_{\mathcal{U}} d(x_n,y_n) = 0.$$

The limit over $\mathcal{U}$ exists since M is bounded. Then we consider the quotient set $\widetilde{M}$. An element $\tilde{x} \in \widetilde{M}$ is a subset of $\mathcal{M}$. If $(x_n) \in \tilde{x}$, then $(y_n) \in \tilde{x}$ if and only if $\lim_{\mathcal{U}} d(x_n, y_n) = 0$. On $\widetilde{M}$ define the metric $\tilde{d}$ by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{\mathcal{U}} d(x_n, y_n)$$

where (x_n) (resp. (y_n)) is any element in $\tilde{x}$ (resp. $\tilde{y}$). It is easy to see that $\tilde{d}$ defines a metric on $\widetilde{M}$ which has many nice properties similar to the linear ultrapower of a Banach space. For example, it is obvious that M is isometric to a subset of $\widetilde{M}$. Indeed, let

$$\dot{M} = \{\widetilde{(x_n)}; x_n = x \text{ for any } n \geq 1\} \ .$$

Then it is easy to show that M and $\dot{M}$ are isometric. In the sequel we will use the notation $M = \dot{M}$ and see $x \in M$ as an element of $\widetilde{M}$ as well. Also it is worth to mention that if M is complete, then $\widetilde{M}$ is complete. The proof is similar to the linear one. In the linear case it is known that if X is a finite dimensional Banach space, then $\widetilde{X}$ is also a finite dimensional Banach space with the same dimension as X. The analogue of this is the following result.

Proposition 1. *If M is compact then $\widetilde{M}$ is also compact and is isometric to M.*

Proof. Since M is compact, then for any sequence $(x_n) \in M$ the limit $\lim_{\mathcal{U}} x_n = x$ exists (in M) since $\mathcal{U}$ is an ultrafilter. So we have

$$\lim_{\mathcal{U}} d(x_n, x) = 0$$

or equivalently $\widetilde{(x_n)} = x$. Hence $\widetilde{M}$ is a subset of $\dot{M}$, i.e. $\widetilde{M} = \dot{M}$. Therefore $\widetilde{M}$ is isometric to M and must be compact. $\square$

Clearly one may then ask what if M is not compact. In this case it is natural to use measures of noncompactness. The most commonly used were introduced by Hausdorff and Kuratowski (see [ADL] for more).

Definition 1. Let (M, d) be a metric space and let $\mathcal{B}(M)$ denote the collection of nonempty, bounded subsets of M.

(1) *The Kuratowski measure of noncompactness* $\alpha : \mathcal{B}(M) \to [0, \infty)$ is defined by

$$\alpha(A) = \inf\{\varepsilon > 0;\ A \subset \bigcup_{i=1}^{i=n} A_i \ \text{ with } A_i \in \mathcal{B}(M) \text{ and } diam(A_i) \leq \varepsilon\} \ .$$

(2) *The Hausdorff (or ball) measure of noncompactness* $\chi : \mathcal{B}(M) \to [0, \infty)$ is defined by

$$\chi(A) = \inf\{r > 0;\ A \subset \bigcup_{i=1}^{i=N} B(x_i, r) \ \text{ with } x_i \in M \ \} \ ,$$

where $B(x, r)$ denote the closed ball centered at x with radius r.

We have the following more general result:

Proposition 2. *Let A be a bounded subset of M. Set $\widetilde{A} = \{\widetilde{(x_n)};\ x_n \in A\}$. Then we have*

$$\chi(A) = \chi(\widetilde{A})\ .$$

Proof. Let $\varepsilon > \chi(A)$ and $\delta > 0$. Then by definition of χ, there exists a finite set $D = \{x_1, x_2, \cdots, x_N\}$ such that

$$A \subset \bigcup_{i=1}^{i=N} B(x_i, \varepsilon)\ .$$

Consider $\widetilde{D}$. Our previous result implies that $\widetilde{D}$ is compact. So there exists a finite set $\{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_K\}$ such that

$$\widetilde{D} \subset \bigcup_{i=1}^{i=K} B(\tilde{x}_i, \delta)\ .$$

From here it is easy to see that

$$\widetilde{A} \subset \bigcup_{i=1}^{i=K} B(\tilde{x}_i, \varepsilon + \delta)$$

which implies

$$\chi(\widetilde{A}) \leq \varepsilon + \delta\ .$$

Since δ was chosen arbitrarily then we have $\chi(\widetilde{A}) \leq \varepsilon$, which implies

$$\chi(\widetilde{A}) \leq \chi(A)\ .$$

In order to complete our proof, we need to show that $\chi(A) \leq \chi(\widetilde{A})$. Let $\varepsilon > 0$. Set $r = \chi(\widetilde{A}) + \varepsilon$. Then there exist $\tilde{x}_i$ $(i = 1, \cdots, K)$ in $\tilde{A}$ such that

$$\widetilde{A} \subset \bigcup_{i=1}^{i=K} B(\tilde{x}_i, r)\ .$$

Set $\tilde{x}_i = \widetilde{(x_i(n))}$, for $i = 1, \cdots, K$, with $x_i(n) \in A$. We claim that for any $\delta > 0$ there exists $n_0 \geq 1$ such that

$$A \subset \bigcup_{i=1}^{i=K} B(x_i(n_0), r + \delta)\ .$$

Assume not. Then there exists $\delta_0 > 0$ such that for any $n \geq 1$, there exists $x_n \in A$ which satisfies

$$x_n \notin \bigcup_{i=1}^{i=K} B(x_i(n), r + \delta_0)\ .$$

Set $\tilde{x} = \widetilde{(x(n))} \in \tilde{A}$. Then

$$\tilde{d}(\tilde{x}, \tilde{x}_i) = \lim_{\mathcal{U}} d(x_n, x_i(n)) \geq r + \delta_0$$

for $i = 1, \cdots, K$. Clearly we have

$$\tilde{x} \notin \bigcup_{i=1}^{i=K} B(\tilde{x}_i, r)$$

which is our desired contradiction. So let $\delta > 0$, we know that there exists $n_0 \geq 1$ such that

$$A \subset \bigcup_{i=1}^{i=K} B(x_i(n_0), r + \delta) \ .$$

This clearly implies

$$\chi(A) \leq r + \delta = \chi(\widetilde{A}) + \varepsilon + \delta \ .$$

Since ϵ and δ were chosen arbitrarly positive, we conclude that

$$\chi(A) \leq \chi(\widetilde{A}) \ ,$$

which completes the proof of our proposition. □

When M is not compact, more can be said about the ultrapower.

Proposition 3. *Assume M is not compact. Then $\widetilde{M}$ is not separable*

Proof. Assume M is not compact. Then there exists a bounded sequence (x_n) with no convergent subsequence. In particular $\lim\limits_{\mathcal{U}} x_{\phi(n)}$ does not exists for any subsequence $(x_{\phi(n)})$ of (x_n). Moreover we can assume that there exists $\varepsilon > 0$ such that

$$\text{sep}(x_n) = \inf\{d(x_n, x_m);\ n \neq m\} \geq \varepsilon \ .$$

For any subsequence $(x_{\phi(n)})$ of (x_n) set $\tilde{x}_\phi = \widetilde{(x_{\phi(n)})}$. Clearly we have

$$\tilde{d}(\tilde{x}_\phi, \tilde{x}_\alpha) = \lim_{\mathcal{U}} \tilde{d}(x_{\phi(n)}, x_{\alpha(n)}) \geq \varepsilon \ .$$

Since any sequence has uncountably many subsequences, the above result implies that $\widetilde{M}$ has an uncountable ε-separated set. Therefore $\widetilde{M}$ is not separable. □

It is quite an amazing result since a linear version of it is also known.

Next we discuss how Lipschitzian mappings extend naturally to the ultrapower. Indeed, let $T : M \to M$ be a Lipschitzian mapping with L as a constant of Lipschitz, i.e.

$$d(T(x), T(y)) \leq L\ d(x, y) \ \text{ for } x, y \in M.$$

Then

$$\lim_{\mathcal{U}} d(x_n, y_n) = 0 \ \text{ implies } \ \lim_{\mathcal{U}} d(T(x_n), T(y_n)) = 0 \ .$$

This obviously implies that $\tilde{T} : \widetilde{M} \to \widetilde{M}$ defined by

$$\tilde{T}(\widetilde{(x_n)}) = (\widetilde{T(x_n)})$$

is well defined. It is easy to check that $\tilde{T}$ is Lipschitzian with L as a constant of Lipschitz. We also have

$$\tilde{T}(x) = T(x) \ \text{ for any } x \in M \ .$$

Before we jump to the next section where the main result of this work will be stated, it is worth to mention that hyperconvexity is not a super-property, i.e. the ultrapower of a hyperconvex metric space is not necessarily hyperconvex.

For more on ultrapowers and nonstandard techniques, the interested reader is advised to consult [AK] and [Sm].

Main Result

Before we state the main result of this work, we will need some definitions. Let M be a metric space. A map $T : M \to M$ is said to be *asymptotically nonexpansive* if there exists a sequence of positive numbers $\{k_n\}$, with $\lim\limits_{n\to\infty} k_n = 1$, such that

$$d(T^n(x), T^n(y)) \leq k_n d(x, y) \ \text{ for any } x, y \in M \text{ and } n = 1, 2, \cdots \ .$$

The main result of our work goes as follows:

Theorem. *Let H be a bounded hyperconvex metric space and $T : H \to H$ be asymptotically nonexpansive mapping. Then T has approximate fixed points, i.e.*

$$\inf\{d(x, T(x));\ x \in H\} = 0 \ .$$

Proof. In order to prove the above conclusion, we need to show that for any $\varepsilon > 0$, there exists $x \in H$ such that

$$d(x, T(x)) \leq \varepsilon.$$

Using the metric convexity of H, we define the map

$$T_n = \frac{1}{k_n} T^n \oplus \left(1 - \frac{1}{k_n}\right) x_0$$

where x_0 is a fixed point in H and k_n is the Lipschitz constant of T^n. The maps $\{T_n\}$ are nonexpansive. Consider the ultrapower $\widetilde{H}$ of H, over a nontrivial ultrafilter $\mathcal{U}$. Define the operators $\hat{T}$ and $\tilde{T}$ by

$$\hat{T}(\tilde{x}) = \hat{T}(\widetilde{(x_n)}) = (\widetilde{T^n(x_n)}) \ \text{ and } \ \tilde{T}(\tilde{x}) = \tilde{T}(\widetilde{(x_n)}) = (\widetilde{T(x_n)}) \ .$$

Since T is asymptotically nonexpansive mapping, the map $\hat{T}$ is nonexpansive. Moreover we have

$$\hat{T}(\widetilde{(x_n)}) = (\widetilde{T_n(x_n)})\ .$$

Since T_n is nonexpansive, Sine and Soardi's fixed point theorem implies the existence of a fixed point x_n (of T_n). The point $\tilde{x} = \widetilde{(x_n)}$ is a fixed point of $\hat{T}$. Hence the fixed point set $Fix(\hat{T})$ is a nonempty subset of $\widetilde{H}$. Since the two operators $\hat{T}$ and $\tilde{T}$ commute, then $\tilde{T}$ leaves invariant the set $Fix(\hat{T})$. It is easy to show that $\tilde{T}$ restricted to $Fix(\hat{T})$ is in fact an isometry (in particular it is nonexpansive). Fix $\varepsilon > 0$. Let $\tilde{x}_i \in Fix(\hat{T})$, i=1,..,N. If $\tilde{x}_i = \widetilde{\Big(x_n(i)\Big)}$, for $i = 1, \cdots, N$, set

$$\varepsilon_n = \max_{1 \leq i \leq N} d\Big(x_n(i), T_n(x_n(i))\Big)\ .$$

Then we have

$$\lim_{\mathcal{U}} \varepsilon_n = 0\ .$$

Set

$$H_n = \{x \in H;\ d(x, T_n(x)) \leq \varepsilon_n\}\ .$$

Then $H_n \neq \emptyset$ because $x_n(i) \in H_n$, for i=1,..,N. Since T_n is nonexpansive, Sine [Sn2] proved that H_n is hyperconvex. Therefore, there exists

$$z_n(i) = \varepsilon x_n(1) \oplus (1-\varepsilon) x_n(i)\ \in H_n$$

for i = 1,..,N. Consider, the point

$$\tilde{z}_i = \widetilde{\Big(z_n(i)\Big)}$$

which we will denote $\varepsilon\tilde{x}_1 \oplus (1-\varepsilon)\tilde{x}_i$. Then we have, $\tilde{z}_i \in Fix(\hat{T})$, and

$$d(\tilde{z}_i, \tilde{z}_j) \leq (1-\varepsilon) d(\tilde{x}_i, \tilde{x}_j)$$

for i,j=2,..,N. Back to our maps $\hat{T}$ and $\tilde{T}$. Let $\tilde{x} \in Fix(\hat{T})$. Write $\tilde{x} = \tilde{x}_1$. Then from the above ideas, there exists $\tilde{x}_2 \in Fix(\hat{T})$ such that

$$\tilde{x}_2 = \varepsilon\tilde{x}_1 \oplus (1-\varepsilon)\tilde{T}\Big(\tilde{x}_1\Big)\ .$$

By induction, we will construct a sequence $\Big(\tilde{x}_n\Big)$ of points in $Fix(\hat{T})$ defined by

$$\tilde{x}_{n+1} = \varepsilon\tilde{x}_1 \oplus (1-\varepsilon)\tilde{T}\Big(\tilde{x}_n\Big)\ .$$

We have for any $n < m$,

$$d(\tilde{x}_n, \tilde{x}_m) \leq (1-\varepsilon) d\left(\tilde{T}\Big(\tilde{x}_{n-1}\Big), \tilde{T}\Big(\tilde{x}_{m-1}\Big)\right)$$

and since $\tilde{T}$ is nonexpansive when restricted to $Fix(\hat{T})$, we get

$$d(\tilde{x}_n, \tilde{x}_m) \leq (1-\varepsilon) d\Big(\tilde{x}_{n-1}, \tilde{x}_{m-1}\Big) \,.$$

This clearly implies that the sequence $\Big(\tilde{x}_n\Big)$ is a Cauchy sequence. Hence it converges to $\tilde{\omega} \in Fix(\hat{T})$. Moreover we have

$$d\Big(\tilde{\omega}, \tilde{T}(\tilde{\omega})\Big) = \lim_{n\to\infty} d\Big(\tilde{x}_{n+1}, \tilde{T}(\tilde{x}_n)\Big) \leq \varepsilon \lim_{n\to\infty} d\Big(\tilde{x}_1, \tilde{T}(\tilde{x}_n)\Big) \,.$$

If we set $\delta = \text{diameter}(H)$, we get

$$d\Big(\tilde{\omega}, \tilde{T}(\tilde{\omega})\Big) \leq \varepsilon\delta \,.$$

Therefore, we have proved that for any $\varepsilon > 0$, there exists $\tilde{\omega}_\varepsilon \in Fix(\hat{T})$, such that

$$d\Big(\tilde{\omega}_\varepsilon, \tilde{T}(\tilde{\omega}_\varepsilon)\Big) \leq \varepsilon \,.$$

From this it is easy to extract $x_\varepsilon \in H$, such that

$$d\Big(x_\varepsilon, T(x_\varepsilon)\Big) \leq \varepsilon$$

for any $\varepsilon > 0$. $\square$

Remark. *Recall that Kirk's original question was about the existence of a fixed point for such mappings. This problem is still open. But one may use a simple embedding of any metric space M into $l_\infty(M)$ (see* [EK] *for more on this), to show that in fact the main problem described here is equivalent to the same problem for the unit ball of l_∞. It is worth to mention that the existence of fixed point for asymptotically nonexpansive mappings is closely related to the existence of fixed point for k-uniformly Lipschitzian mappings in the linear case (see* [KX] *for more on this).*

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