ON THE NUMERICAL INDEX OF VECTOR-VALUED FUNCTION SPACES

Elmouloudi ED-DARI, Mohamed Amine KHAMSI, and Asuman Güven AKSOY

Abstract. Let $X$ be a Banach space and $\mu$ a positive measure. We show that $n(L_p(\mu, X)) = \lim_{m \to \infty} n(l_p^m(X)), 1 \leq p < \infty$. Also we investigate the positivity of the numerical index of $l_p$-spaces.

1 Introduction.

Let $X$ be a Banach space over IR or IC, we write $B_X$ for the closed unit ball and $S_X$ for the unit sphere of $X$. The dual space is denoted by $X^*$ and the Banach algebra of all continuous linear operators on $X$ is denoted by $B(X)$. The numerical range of $T \in B(X)$ is defined by $V(T) = \{x^*(Tx) : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$. The numerical radius of $T$ is then given by $v(T) = \sup\{||\lambda|| : \lambda \in V(T)\}$. Clearly, $v$ is a semi norm on $B(X)$ and $v(T) \leq ||T||$ for all $T \in B(X)$. The numerical index of $X$ is defined by $n(X) = \inf\{v(T) : T \in S_{B(X)}\}$.

The concept of numerical index was first suggested by Lumer [7] in 1968. Since then a lot of attention has been paid to this constant of equivalence between the numerical radius and the usual norm in the Banach algebra of all bounded linear operators of a Banach space. Classical references here are [1], [2]. For recent results we refer the reader to [3], [5], [6], [8], [10].

In this paper we show that for any positive measure $\mu$ and Banach space $X$, the numerical index of $L_p(\mu, X)$, $1 \leq p < \infty$ is the limit of the sequence of numerical index of $l_p^m(X)$. This gives a partial answer to Martín’s question [9] and generalizes the result obtained for the scalar case [5]. Also we study the positivity of the numerical index of $l_p$-space.

Mathematics Subject Classification (2000): 47A12.

Key words: Numerical index - Numerical radius.
Here $L_p(\mu, X)$ is the classical Banach space of $p$-integrable functions $f$ from $\Omega$ into $X$ where $(\Omega, \Sigma, \mu)$ is a given measure space. And $l_p(X)$ is the Banach space of sequences $x = (x_n)_{n \geq 1}$, $x_n \in X$, such that $\sum_{n=1}^{\infty} ||x_n||^p < \infty$. And finally $l^m_p(X)$ is the Banach space of finite sequences $x = (x_n)_{1 \leq n \leq m}$, $x_n \in X$, equipped with the norm $||x|| = \left( \sum_{n=1}^{m} ||x_n||^p \right)^{\frac{1}{p}}$.

2 Main results.

Theorem 2.1. Let $X$ be a Banach space. Then, for every real number $p, 1 \leq p < \infty$, the numerical index of the Banach space $l_p(X)$ is given by

$$n(l_p(X)) = \lim_{m \to \infty} n(l^m_p(X)).$$

Proof. Let $m \geq 1$ and $T : l^m_p(X) \to l^m_p(X)$ $x \mapsto (T_1(x), ..., T_m(x))$. Define the linear operator $\tilde{T} : l_p(X) \to l_p(X)$ as follows for $x = (x_1, ..., x_m, x_{m+1}, ...) \in l_p(X)$, $\tilde{T}(x) = (T_1(x_1, ..., x_m), ..., T_m(x_1, ..., x_m), 0, ...)$ Clearly, $\tilde{T}$ is bounded and $||T|| = ||\tilde{T}||$. We have also $v(T) = v(\tilde{T})$. To prove this, let us first note that if $x = (x_1, ..., x_m, ...) \in S_{l_p(X)}$, then there exists an element, namely $x_k^*$, in $S_{l^*(X^*)}$, where $q$ is the conjugate exponent to $p$, such that $x_k^*(x) = 1$. Explicitly $x_k^* = (||x_1||^{p-1}x_1^*, ..., ||x_m||^{p-1}x_m^*, ...)$ where the $x_k^*$’s are taken in $S_X^*$ such that $x_k^*(x_k) = ||x_k||$. Now, let $\varepsilon > 0$. Following the expression $v(\tilde{T}) = \sup\{|x_k^*(\tilde{T}x)| : x \in S_{l_p(X)}\}$ (cf. [4], Lemma 3.2 and Proposition 1.1) there exists $x = (x_1, ..., x_m, x_{m+1}, ...) \in S_{l_p(X)}$ such that

$$v(\tilde{T}) - \varepsilon < |x_k^*(\tilde{T}x)| = ||||x_1||^{p-1}x_1^*, ..., ||x_m||^{p-1}x_m^*(T(x_1, ..., x_m))|.$$

Put $r := \left( \sum_{k=1}^{m} ||x_k||^p \right)^{1/p} \leq 1$. Then we obtain $v(\tilde{T}) - \varepsilon < r^p v(T)$ which yields $v(\tilde{T}) \leq v(T)$. The reverse inequality is easy. Therefore

$$\{v(T) : T \in l^m_p(X), ||T|| = 1\} \subset \{v(U) : U \in l_p(X), ||U|| = 1\}$$

which yields $n(l_p(X)) \leq n(l^m_p(X))$. Consequently $n(l_p(X)) \leq \liminf n(l^m_p(X))$. Now we shall prove that $\limsup n(l^m_p(X)) \leq n(l_p(X))$. Let $T \in B(l_p(X))$. Define the sequence of operators $\{S_m\}_m$ as follows; for each $m \geq 1$, $S_m$ is defined on $l^m_p(X)$ by

$$S_m(x) = (T_1(x_1, ..., x_m, 0, 0, ...), ..., T_m(x_1, ..., x_m, 0, 0, ...)) \quad (x \in l^m_p(X)).$$

Clearly, the $S_m$’s are bounded and $||S_m|| \leq ||T||$ for all $m$. We claim that

(i) $||S_m|| \to ||T||$

(ii) $v(S_m) \to v(T)$. 

2
Indeed, we consider the sequence of operators \( \{\tilde{S}_m\}_m \) defined on \( l_p(X) \) by

\[
\tilde{S}_m(x) = (T_1(x_1, ..., x_m, 0, 0, ...), ..., T_m(x_1, ..., x_m, 0, 0, ...), 0, 0, ...)
\]

for all \( x = (x_1, ..., x_m, x_{m+1}, ...) \in l_p(X) \). It is easy to see that \( \|S_m\| = \|\tilde{S}_m\| \), and \( \tilde{S}_m \) converges strongly to \( T \). This implies that \( \|T\| \leq \liminf_m \|\tilde{S}_m\| \), and it follows that \( \|S_m\| \to \|T\| \). As in (i) we have also \( v(S_m) = v(\tilde{S}_m) \), so it is enough to prove that \( v(\tilde{S}_m) \to v(T) \). Let \( \varepsilon > 0 \) and fix \( u \in SX \), \( u^* \in S_X^* \) such that \( u^*(u) = 1 \). There exists \( x \in S_{l_p(X)} \) such that

\[
|x^*_n(Tx)| > v(T) - \varepsilon.
\]

For each \( n \geq 1 \), consider

\[
x^n = (x_1, ..., x_{n-1}, \lambda_n u, 0, 0, ...); \quad x^*_n = (\|x_1\|^{p-1}x^*_1, ..., \|x_{n-1}\|^{p-1}x^*_{n-1}, \lambda_n^{p-1}u^*, 0, 0, ...)
\]

where \( \lambda_n = \left( \sum_{k=m}^{\infty} \|x_k\|^p \right)^{1/p} \). Then

\[
x^*_n(x^n) = 1 = \|x^*_n\| = \|x^n\|.
\]

Moreover, \( \|x-x^n\| \to 0 \) and \( \|x^*_x-x^*_n\| \to 0 \) where \( x^*_x = (\|x_1\|^{p-1}x^*_1, ..., \|x_n\|^{p-1}x^*_n, ...) \). It follows that \( x^*_n(Tx^n) \to x^*_x(Tx) \) as \( n \) tends to infinity. Let \( n_0 \geq 1 \) be such that

\[
|x^*_n(Tx^n)| > v(T) - \varepsilon \quad (n \geq n_0).
\]

Since \( \tilde{S}_m \) converges strongly to \( T \), thus for fixed \( n \geq n_0 \), \( x^*_n(\tilde{S}_mx^n) \) converges to \( x^*_n(Tx^n) \) as \( m \) tends to infinity. So there is \( m_0 \geq n \) such that

\[
|x^*_n(\tilde{S}_mx^n)| > v(T) - \varepsilon \quad (m \geq m_0).
\]

This yields \( v(\tilde{S}_m) > v(T) - \varepsilon \) for all \( m \geq m_0 \) and therefore \( v(\tilde{S}_m) \) converges to \( v(T) \) as \( m \) tends to infinity. Now, following (i) and (ii) we have \( n(l_p(X)) \geq \limsup_m n(l_p^m(X)) \). Indeed, for a given \( \varepsilon > 0 \), we find \( T \in S_{lb(l_p(X))} \) such that

\[
n(l_p(X)) + \varepsilon > v(T).
\]

Since \( v(T) = \lim_m v(\tilde{S}_m) \), there exists \( m_0 \) such that

\[
n(l_p(X)) + \varepsilon > v(\tilde{S}_m) \quad (m \geq m_0).
\]

But \( v(\tilde{S}_m) = v(S_m) \geq n(l_p^m(X)) \|S_m\| \), and \( \|S_m\| \to \|T\| = 1 \), so there exists \( k_0 \geq m_0 \) such that

\[
n(l_p(X)) + \varepsilon > n(l_p^m(X))(1 - \varepsilon) \quad (m \geq k_0).
\]

This implies \( n(l_p(X)) \geq \limsup_m n(l_p^m(X)) \) and completes the proof of Theorem 2.1. \( \square \)

It is well known that \( n(\oplus_{\lambda} X_\lambda)_{l_\infty} = \inf_{\lambda \in \Lambda} n(X_\lambda) \) [9]. This shows that, in particular, \( n(l_\infty(X)) = n(X) \) (\( = \lim_m n(l_\infty^m(X)) \)). So, Theorem 2.1 is also valid for \( p = \infty \).
Theorem 2.2. Let \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space. Then, for every Banach space \(X\) and every real number \(p\), \(1 \leq p < \infty\),
\[ n(L_p(\mu, X)) = n(l_p(X)) \]

Proof. Let us first prove that \(n(L_p(\mu, X)) \leq n(l_p(X))\). For this we adapt the proof due to Javier and Martin for the scalar case (not published result). Indeed, if \(\mu\) is not atomic, \(L_p(\mu, X)\) is isometric to \(L_p(\mu, X) \oplus_p L_p(\mu, X)\), so they have the same numerical index. Let \(T = (T_1, T_2) \in B(l_p^2(\mu, X))\) and define the operator \(S\) on \(L_p(\mu, X) \oplus_p L_p(\mu, X)\) by \(S(f_1, f_2)(\omega) = T(f_1(\omega), f_2(\omega))\). One can check easily that \(\|S\| = \|T\|\). Moreover, \(v(T) = v(S)\). Indeed, let \(f_1 = \sum_{i=1}^{n} x_i \frac{1}{\mu(A_i)^{1/p}}\), \(f_2 = \sum_{i=1}^{n} y_i \frac{1}{\mu(B_i)^{1/p}}\) be simple functions in \(L_p(\mu, X)\) with \(\|(f_1, f_2)\| = \left( \sum_{i=1}^{n} \|x_i\|^p + \sum_{i=1}^{n} \|y_i\|^p \right)^{1/p} = \|T\| = \|S\|\). For each \(i\) we can find \(x_i^*\) and \(y_i^*\) in \(S_X^*\) such that \(x_i^*(x_i) = \|x_i\|^p\) and \(y_i^*(y_i) = \|y_i\|^p\). If we set \(g_1 = \sum_{i=1}^{n} \|x_i\|^{p-1} x_i \frac{1}{\mu(A_i)^{1/q}}\) and \(g_2 = \sum_{i=1}^{n} \|y_i\|^{p-1} y_i \frac{1}{\mu(B_i)^{1/q}}\), we have clearly \((g_1, g_2) \in S_L_{X^*} \oplus L_{L_p(\mu, X^*)}\) and \(\langle (g_1, g_2), (f_1, f_2) \rangle = 1\). Moreover,
\[
\|(g_1, g_2)(S(f_1, f_2))\| \leq \int_{\Omega} \|(g_1(\omega), g_2(\omega))\| d\mu(\omega) \\
\leq v(T) \int_{\Omega} \|f_1(\omega)\|^{p} + \|f_2(\omega)\|^{p} d\mu(\omega) = v(T).
\]
Following [4], we have \(v(S) \leq v(T)\). For the reverse inequality, let \((x_1, x_2) \in S_{l_p^2(X)}\). Take \(A \in \Sigma\) with \(\mu(A) > 0\) and consider \((f_1, f_2) = \left( x_1 \frac{1}{\mu(A)^{1/p}}, x_2 \frac{1}{\mu(A)^{1/p}} \right)\). From what we have just seen \((g_1, g_2) = \left( \|x_1\|^{p-1} x_1 \frac{1}{\mu(A)^{1/q}}, \|x_2\|^{p-1} x_2 \frac{1}{\mu(A)^{1/q}} \right) \in S_{L_{X^*}} \oplus L_{L_p(X^*)}\) and \(\langle (g_1, g_2), (f_1, f_2) \rangle = 1\). Moreover,
\[
\|(g_1(\omega), g_2(\omega))\| = \int_{\Omega} \|(g_1(\omega), g_2(\omega))\|(d\mu(\omega) \leq v(S).
\]
This yields \(v(T) \leq v(S)\). Consequently \(\{v(T) : T \in S_{l_p^2(X)}\} \subseteq \{v(S) : S \in S_{L_{X^*}} \oplus L_{L_p(X^*)}\}\) which yields \(n(L_p(\mu, X) \oplus_p L_p(\mu, X)) \leq n(l_p^2(X))\). So
\[ n(L_p(\mu, X)) \leq n(l_p^2(X)) \]
Now, for any integer \(m \geq 1\), with the same work as above, we obtain
\[ n(L_p(\mu, X)) \leq n(l_p^m(X)) \]
It follows from Theorem 2.1 that
\[ n(L_p(\mu, X)) \leq n(l_p(X)) \]
If \( \mu \) is atomic then \( L_p(\mu, X) \) is isometric to \( L_p(\nu, X) \oplus_p [\oplus_{i \in I} X] \) for a suitable set \( I \) and an atomless measure \( \nu \). With the help of Remark 2 [9], we also have \( n(L_p(\mu, X)) \leq n(l_p(X)) \). The reverse inequality \( n(L_p(\mu, X)) \geq n(l_p(X)) \) follows with the same technique used in [5] for the scalar case.

Corollary 2.3. Let \((\Omega, \Sigma, \mu)\) be a \( \sigma \)-finite measure space. Then, for every Banach space \( X \) and every real number \( p, 1 \leq p < \infty \)

\[
\lim_{m \to \infty} n(l_p^m(X)) = n(l_p(X)).
\]

3 On the positivity of the numerical index of \( l_p \)-space

It was proved that the numerical index of \( l_p^m, p \neq 2, m = 1, 2, \ldots \) cannot be equal to 0 this is equivalent to that the numerical radius and the operator norm are equivalent on \( B(l_p^m), p \neq 2 \) (see Theorem 2.3 [6]). In this section we shall also prove that both norms are equivalent on \( B(l_p, l_p^m) \).

Theorem 3.1. For every real number \( p \geq 1, p \neq 2 \) and every integer \( m \), the numerical radius is equivalent to the operator norm on \( B(l_p, l_p^m) \).

Here \( l_p \) is real and \( l_p^m \) is identified with its natural embedding in \( l_p \).

Proof. Let \( T = (t_{ik}) \in B(l_p, l_p^m) \). We first have

\[
\|T\| \leq \left\| \left( \sum_{k=1}^{\infty} |t_{1k}|^q \right)^{\frac{1}{q}}, \ldots, \left( \sum_{k=1}^{\infty} |t_{mk}|^q \right)^{\frac{1}{q}} \right\|_p
\]

\[
\leq \left( \sum_{k=1}^{\infty} |t_{1k}|^q \right)^{\frac{1}{q}} + \cdots + \left( \sum_{k=1}^{\infty} |t_{mk}|^q \right)^{\frac{1}{q}}.
\]

Consider \( \{T^j\} \in B(l_p, l_p^m) \) defined by \( T^j e_k = T e_k \) for \( k \neq j \) and \( T^j(e_j) = 0 \). Then for \( x = \sum_{k=1}^{\infty} x_k e_k \in S_{l_p} \) we have

\[
x^*_x(T^1 x) = \varepsilon_1 |x_1|^{p-1} \sum_{k=2}^{\infty} t_{2k} x_k + \cdots + \varepsilon_m |x_m|^{p-1} \sum_{k=2}^{\infty} t_{mk} x_k \quad (\varepsilon_j \in \{-1, 1\}).
\]

Take \( x_1 = \varepsilon_1 2^{-1/p} \) with \( \varepsilon_1 \in \{-1, 1\} \) we obtain

\[
|x^*_x(T^1 x)| = 2^{-1/q} \left( \sum_{k=2}^{\infty} t_{1k} x_k \right)^{\varepsilon_1} \left( \sum_{k=2}^{\infty} t_{2k} x_k + \cdots + \varepsilon_m |x_m|^{p-1} \sum_{k=2}^{\infty} t_{mk} x_k \right) \leq v(T^1)
\]

Since \( \varepsilon_1 \) is arbitrary in \( \{-1, 1\} \) then

\[
2^{-1/q} \sum_{k=2}^{\infty} t_{1k} x_k \} + \left| \varepsilon_2 |x_2|^{p-1} \sum_{k=2}^{\infty} t_{2k} x_k + \cdots + \varepsilon_m |x_m|^{p-1} \sum_{k=2}^{\infty} t_{mk} x_k \right| \leq v(T^1).
\]
And in particular

\[ 2^{-1/q} \sum_{k=2}^{\infty} t_{1k} x_k \leq v(T^1) \]

for all \((x_2, \ldots, x_m, \ldots) \in l_p\) such that \(\sum_{k=2}^{\infty} |x_k|^p = \frac{1}{2}\). That is

\[ \frac{1}{2} \left| \sum_{k=2}^{\infty} t_{1k} y_k \right| \leq v(T^1) \quad \forall (y_2, \ldots, y_m, \ldots) \in S_{l_p} \]

which yields

\[ \frac{1}{2} \left( \sum_{k \neq 1} |t_{1k}|^q \right)^{\frac{1}{q}} \leq v(T^1). \]

The same work as above shows that

\[ \frac{1}{2} \left( \sum_{k \neq j} |t_{jk}|^q \right)^{\frac{1}{q}} \leq v(T^j) \quad \text{(\star)} \]

for \(j = 1, 2, \ldots, m\). Now let \(R^j = T - T^j\) then we have

\[ v(T^j) \leq v(T) + \|R^j\|. \]

And following (\star) we obtain

\[ \left( \sum_{k=1}^{\infty} |t_{jk}|^q \right)^{\frac{1}{q}} \leq 2 \left( v(T) + \|R^j\| \right) + |t_{jj}| \]

which yields

\[ \|T\| \leq 2mv(T) + 2 \sum_{j=1}^{m} \|R^j\| + \sum_{j=1}^{m} |t_{jj}|. \]

Now let \(\{T_n\}\) be a \(v\)-cauchy sequence in \(B(l_p, l^m_p)\). Since \(v(T_nP_m) = v(P_mT_nP_m) \leq v(T_n)\) where \(P_m\) is the operator projection on \(l^m_p\) (see [5] p 4), and using the fact that in finite dimensional space \(l^m_p\) both norms are equivalent, then each \(R^j_n = T_n - T^j_n\) converges in operator norm to some \(R^j\). Therefore \(\{T_n\}\) is \(\|\|\)-cauchy. This completes the proof of the Theorem 3.1. \(\square\)

It’s still unknown if the numerical radius and the operator norm are equivalent on the Banach space \(B(l_p), p \neq 2\) which gives a complete answer to the question of C. Finet and D. Li.
References


E. ED-DARI
Université d’Artois
Faculté des Sciences Jean Perrin
SP 18
62307-Lens Cedex
FRANCE
Elnouldi.Eddari@euler.univ-artois.fr

M. A. KHAMSIP
University of Texas at El Paso
Dept. of Mathematical Sciences
500 West University
Ave. El Paso, Texas 79968-0514
USA
mohamed@math.utep.edu

A. G. AKSOY
Department of Mathematics
Claremont McKenna College
Claremont, CA 91711
asuman.aksoy@claremontmckenna.edu