On the existence of minimal elements in partially ordered sets.

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Abstract

In this work, we give a characterization of the existence of minimal elements in partially ordered sets in terms of fixed point type statement. This characterization shows that the assumptions in Caristi's fixed point theorem can, a priori, be weakened. Finally, we give a negative answer to Kirk’s problem on an extension of Caristi’s result.
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1 Introduction

This work was motivated by a problem stated by Kirk [5] to improve the Caristi’s fixed point theorem [1,5]. Recall that this theorem states that any map $T : M \to M$ defined on a complete metric space has a fixed point provided that there exists a lower semicontinuous map $\phi$ mapping $M$ into the nonnegative numbers such that

\[
(*) \quad d(x, Tx) \leq \phi(x) - \phi(Tx)
\]

for every $x \in M$. This general fixed point theorem has found many applications in nonlinear analysis. It is shown, for example, that this theorem yields essentially all the known inwardness results [7] of geometric fixed point theory in Banach spaces. Recall that inwardness conditions are the ones which assert that, in some sense, points from the domain are mapped toward the domain. Possibly the weakest of the inwardness conditions, the Leray-Schauder boundary condition is the assumption that a map points $x$ of $\partial M$ anywhere except to the outward part of the ray originating at some interior point of $M$ and passing through $x$.

The proofs given to Caristi’s result vary and use different techniques (see [1,3,4,6,8]). In this note we prove a characterization to the existence of minimal elements in partially ordered sets in terms of fixed point. Then we show how Caristi’s theorem can be deduced from this result.

2 Main results

Let $A$ be an abstract set partially ordered by $\prec$. We will say that $a \in A$ is a minimal element of $A$ if and only if $b \prec a$ implies $b = a$.

Theorem 1. Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent.

1. $A$ contains a minimal element,
2. Any set valued map $T$ defined on $A$ such that $y \prec x$ for any $x \in A$ and $y \in Tx$, has a fixed point, i.e there exists $a$ in $A$ such that $a \in Ta$. 

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Proof (1) ⇒ (2) Obviously any minimal element is fixed by \( T \). We complete the proof of Theorem 1 by showing that (2) ⇒ (1). Assume that \( A \) fails to have a minimal element. Define the set valued map \( T \) on \( A \) by

\[
T(x) = \{ y \in A; \ y \prec x \ with \ y \neq x \},
\]

for any \( x \in A \). Clearly our assumption on \( A \) implies that \( T(x) \) is not empty for any \( x \in A \). (2) will imply that \( T \) has a fixed point \( a \in A \). Contradiction with the definition of \( T \). So, the proof of Theorem 1 is complete.

Remark Recall that Taskovic [9] showed that Zorn’s lemma is equivalent to:

(TT) Let \( \mathcal{F} \) be a family of selfmappings defined on a partially ordered set \( A \) such that

\[
x \leq f(x) \ (\text{resp.} \ f(x) \leq x),
\]

for all \( x \in A \) and all \( f \in \mathcal{F} \). If each chain in \( A \) has an upper bound (resp. lower bound), then the family \( \mathcal{F} \) has a common fixed point.

So, Theorem 1 is different from the above result since in our statement we consider the existence of minimal elements, which in general does not imply that any linearly ordered subset has a lower bound.

In the next result we discuss a common fixed point theorem. Let \((M, d)\) be a metric space and \( \phi : M \to [0, \infty) \) be a map. Define the order \( \prec\phi \) (see [1,2]) on \( M \) by

\[
x \prec\phi y \ iff \ d(x, y) \leq \phi(y) - \phi(x),
\]

for any \( x,y \) in \( M \). It is straightforward that \((M, \prec\phi)\) is a partially ordered set. However it is not clear what are the minimal assumptions on \( M \) and \( \phi \) which oblige \( M \) to have minimal elements. In particular, if \( M \) is complete and \( \phi \) is lower semicontinuous, then any decreasing chain in \((M, \prec\phi)\) has a lower bound. Indeed, let \((x_\alpha)_{\alpha \in \Gamma}\) be a decreasing chain, then \((\phi(x_\alpha))_{\alpha \in \Gamma}\) is a decreasing net of positive numbers. Let \((\alpha_n)\) be an increasing sequence of elements from \( \Gamma \) such that

\[
\lim_{n \to \infty} \phi(x_{\alpha_n}) = \inf\{\phi(x_{\alpha}); \alpha \in \Gamma\}.
\]
Using the definition of $\prec_\phi$ one can easily show that $(x_{\alpha_n})$ is Cauchy and therefore converges to $x \in M$. Finally, it is straightforward that $x \prec_\phi x_\alpha$ for all $\alpha \in \Gamma$, which means that $x$ is a lower bound for $(x_\alpha)_{\alpha \in \Gamma}$. Zorn’s lemma will therefore imply that $(M, \prec_\phi)$ has minimal elements.

**Corollary 1.** Let $(M, \prec_\phi)$ be as described above. Assume that $a \in M$ is a minimal element. Then, any map $T : M \to M$ such that for all $x \in M$

$$d(x, Tx) \leq \phi(x) - \phi(Tx),$$

(i.e, $Tx \prec_\phi x$) fixes $a$, i.e, $Ta = a$.

**Remark.** This corollary can be seen as a generalization of Caristi’s result. Indeed, the regular assumptions made in Caristi’s theorem imply that any linearly ordered subset (for $\prec_\phi$) has a lower bound, which is stronger than having a minimal element (see the remark following Theorem 1.).

Using Corollary 1 and the discussion above, we get the following result which can be seen as a common fixed point theorem [2,9].

**Corollary 2.** Let $M$ be a complete metric space, and $\phi : M \to [0, \infty)$ be a lower semicontinuous function. Let $(T_i)_{i \in I}$ be a family of selfmaps defined on $M$ such that

$$d(x, T_i(x)) \leq \phi(x) - \phi(T_i(x)),$$

for all $i \in I$. Then, any minimal element of $(M, \prec_\phi)$ is a common fixed point of the family $(T_i)_{i \in I}$.

In attempting to improve Corollary 2, Kirk[5] has raised the question of whether a map $T : M \to M$ such that for all $x \in M$

$$\text{(**) } d(x, Tx)^p \leq \phi(x) - \phi(Tx),$$

for some $p > 1$, has a fixed point. In what follows we give an example of a free fixed point map $T$ which satisfies (**).

**Example.** Let $M = \{x_n : n \geq 1\} \subset [0, \infty)$ defined by

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$
for all $n \geq 1$. Then $M$ is a closed subset of $[0, \infty)$ and therefore is complete. Define $T : M \to M$ by $Tx_n = x_{n+1}$ for all $n \geq 1$. Then,

$$d(x, Tx)^p = \frac{1}{(n + 1)^p} = \phi(x) - \phi(Tx),$$

where $\phi(x_n) = \sum_{n+1 \leq i \leq 1} \frac{1}{i^p}$, for all $n \geq 1$. It is easy to show that $\phi$ is lower semicontinuous. Furthermore one can also show that $T$ is nonexpansive, i.e

$$d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in M$. And it is clear that $T$ fails to have a fixed point.

**Remark.** It is commonly known that any contraction map $T : M \to M$ (i.e $d(Tx, Ty) \leq k \, d(x, y)$ with $k \in (0,1)$, and more generally any map $T$ which satisfies for any $x \in M$

$$d(T^2(x), T(x)) \leq kd(x, Tx) \text{ with } k \in (0,1)$$

satisfies (*) with $\phi(x) = \frac{1}{1-k}d(x, Tx)$. Clearly one can deduce that, in this case, $\phi(Tx) \leq k\phi(x)$ for all $x \in M$.

More generally, let $T$ be a selfmap defined on a complete metric space $M$ which satisfies (**)$^\ast$ and such that $\phi(Tx) \leq k\phi(x)$ holds with $k \in (0,1)$. Then the Picard iterates $(T^n(x))$ (for any $x \in M$) converges to a fixed point. Indeed, one can easily show that $T$ satisfies (*) where the new function $\phi'$ is defined by $\phi'(x) = K\phi^{\frac{1}{r}}(x)$ for every $x \in M$, where $K$ is a constant. So for any $x \in M$ $(T^n(x))$ is a Cauchy sequence, which converges to $a \in M$. Clearly $\phi(a) \leq \liminf \phi(T^n(x))$. And since $\phi(T^n(x)) \leq k^n\phi(x)$, we obtain $\phi(a) = 0$. This obviously implies that $d(a, Ta)^p = 0$, i.e $Ta = a$.

**References**


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