Uniform Smoothness implies Super-Normal Structure Property.

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Abstract

We prove that a uniformly smooth Banach space X has super-normal structure property. More precisely we prove that if the modulus of smoothness of X satisfies $\lim_{\tau\to 0}\dfrac{\rho_X(\tau)}{\tau} < \dfrac{1}{2}$, then X and X^* are superreflexives and have super-normal structure property.

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1 Introduction and Preliminaries.

The results of the present paper are a part of the research done by the author during his Ph.D. at the University of Paris 6, 1987.

The condition on the modulus of smoothness that we consider in this work originated in Baillon's work [3] where he proved that if $\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} < \frac{1}{2}$, then the Banach space X has the fixed point property. This result was strengthened by Turett [14] showing that this condition on the modulus of smoothness implies normal structure property and therefore, via Kirk's theorem [9], implies the fixed point theorem.
In this work we give an easy proof of Turett's result and show that under the cited condition the Banach space and its dual enjoy super-normal structure property. We also notice that super-normal structure property implies uniform normal structure property.

To our knowledge this is the only proof of this fact that uses nonstandard methods.

Throughout this work $(X, ||.||)$ will be a Banach space. We will denote by S_X its unit sphere, i.e. $S_X = \{x \in X; ||x|| = 1\}$, and by X^* its dual space. We begin with some standard definitions.

Definition 1. Let X be a Banach space.

(a) Define for every $x, y \in S_X$ and every $\tau \in [0, 1]$

$$\rho_X(x, y, \tau) = \frac{1}{2}(||x + \tau y|| + ||x - \tau y||) - 1,$$

(b) and

$$\rho_X(y, \tau) = \sup_{x \in S_X} \rho_X(x, y, \tau).$$

(c) Define the modulus of smoothness of X by

$$\rho_X(\tau) = \sup_{y \in S_X} \rho_X(y, \tau).$$

(d) We will say that X is uniformly smooth if and only if

$$\lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} = 0.$$

Definition 2. Let X be a Banach space.

(a) Define for every $z \in S_X$ and every $\epsilon \in [0,2]$

$$\delta_X(\epsilon, z) = \inf\{1 - ||\frac{x+y}{2}||;\ (x,y) \in S_X \ \ \textit{such that} \ \ x - y = \alpha z \ \ \textit{with} \ \ |\alpha| \geq \epsilon\}.$$

(b) Define the modulus of uniform convexity of X by

$$\delta_X(\epsilon) = \inf\{\delta_X(\epsilon, z);\ z \in S_X\}.$$

(c) Define the characteristic of uniform convexity of X in the direction $z \in S_X$ to be

$$\epsilon_0(X, z) = \sup\{\epsilon;\ \delta_X(\epsilon, z) = 0\},$$

and the characteristic of uniform convexity to be

$$\epsilon_0(X) = \sup\{\epsilon;\ \delta_X(\epsilon) = 0\}.$$

(d) We will say that X is uniformly convex if and only if $\delta_X(\epsilon) > 0$ for every $\epsilon \in [0,2]$ (i.e. $\epsilon_0(X) = 0$).

Recall that these two notions are dual to each other. Indeed in [14] one can find the proof of the following technical result.

Lemma 1. For every Banach space X we have

(i) $\rho_{X^*}(\tau) = \sup\{\frac{\tau\epsilon}{2} - \delta_X(\epsilon);\ \epsilon \in [0,2]\}$, for $\tau > 0$.

(ii) $\rho_X(\tau) = \sup\{\frac{\tau\epsilon}{2} - \delta_{X^*}(\epsilon);\ \epsilon \in [0,2]\}$, for $\tau > 0$.

(iii) X is uniformly convex if and only if X^* is uniformly smooth.

The characteristic of convexity is used to scale Banach spaces. For example James [7] proved that if X is uniformly nonsquare (i.e. $\epsilon_0(X) < 2$) then X is superreflexive (see [13] for more details). Also it is well known that the condition $\epsilon_0(X) < 1$ implies uniform normal structure (see for example [6]). In the following technical result we compare the characteristic of convexity to $\lim\limits_{\tau\to 0}\frac{\rho_X(\tau)}{\tau} = \rho'_X(0)$. This limit will therefore be used to scale Banach spaces from smoothness point of view.

<u>**Theorem 1.**</u> For every Banach space X we have

$$(1) \quad \lim_{\tau\to 0}\frac{\rho_X(\tau)}{\tau} < \frac{\alpha}{2} \;\; iff \;\; (2) \quad \epsilon_0(X^*) < \alpha,$$

for every $\alpha \le 2$.
Therefore, we have

$$\lim_{\tau\to 0}\frac{\rho_X(\tau)}{\tau} = \frac{1}{2}\epsilon_0(X^*).$$

<u>Proof.</u> Let us first show that (1) implies (2). Let $\alpha \in [0, 2]$ and assume that $\epsilon_0(X^*) \ge \alpha$. Then there exist (f_n) and (g_n) in S_{X^*} such that

$$||f_n - g_n||_{X^*} \ge \alpha \;\; and \;\; \lim_{n\to\infty}||f_n + g_n||_{X^*} = 2.$$

On the other hand using the definition of ρ_X we get

$$\rho_X(\tau) \ge ||\frac{x+\tau y}{2}|| + ||\frac{x-\tau y}{2}|| - 1,$$

for every $\tau > 0$ and $x, y \in S_X$. Therefore

$$\rho_X(\tau) \ge |\frac{f(x)+g(x)}{2}| + \tau|\frac{f(y)-g(y)}{2}| - 1,$$

for every $f, g \in S_{X^*}$. Since x and y were arbitrary we get

$$\rho_X(\tau) \ge ||\frac{f+g}{2}|| + \tau||\frac{f-g}{2}|| - 1.$$

So in particular we have for every $n \ge 1$

$$\rho_X(\tau) \ge ||\frac{f_n+g_n}{2}|| + \tau||\frac{f_n-g_n}{2}|| - 1.$$

Let n goes to infinity, we obtain

$$\rho_X(\tau) \geq \frac{\tau\alpha}{2}.$$

This clearly means that (1) implies (2). Let us complete the proof by showing that (2) implies (1). So assume that $\epsilon_0(X^*) < \alpha$ and let $\alpha' \in (\epsilon_0(X^*), \alpha)$. Set $\tau' = \delta_{X^*}(\alpha')$ and consider $\epsilon \in [0,2]$. Two cases occur.
First assume $\epsilon < \alpha'$ then $\frac{\tau\epsilon}{2} < \frac{\tau\alpha'}{2}$. So $\frac{\tau\epsilon}{2} - \delta_{X^*}(\epsilon) < \frac{\tau\alpha'}{2}$.
On the other hand if $\alpha' \leq \epsilon$ then $\delta_{X^*}(\epsilon) \geq \delta_{X^*}(\alpha') = \tau'$, since the modulus of convexity is an increasing function. Therefore

$$\frac{\tau\epsilon}{2} \leq \tau < \tau' < \delta_{X^*}(\epsilon)$$

for any $\tau < \tau'$. But this clearly implies

$$\frac{\tau\epsilon}{2} - \delta_{X^*}(\epsilon) < 0.$$

Therefore in any case we have for $\tau < \tau'$

$$\sup\{\frac{\tau\epsilon}{2} - \delta_{X^*}(\epsilon); \epsilon \in [0,2]\} \leq \frac{\tau\alpha'}{2}.$$

Using Lemma 1 we get $\rho_X(\tau) \leq \frac{\tau\alpha'}{2}$. Which clearly implies that

$$\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} \leq \frac{\alpha'}{2}.$$

Our choice of α' implies that (2) is true.
Let us now complete the proof of Theorem 1. Assume first that $\epsilon_0(X^*) = 2$. Then $\delta_{X^*}(\epsilon) = 0$ for every $\epsilon \in [0,2]$. Therefore using Lemma 1 again we get $\rho_X(\tau) = \tau$ for every $\tau > 0$. This will imply

$$\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} = 1 = \frac{\epsilon_0(X^*)}{2}.$$

Now if we assume that $\epsilon_0(X^*) < 2$, then clearly we can use our previous result to get the desired conlusion.

2 Uniform smoothness and normal structure property.

Throughout the sequel the diameter of a subset A of X is denoted by $diam(A)$ ($diam(A) = \sup\{||x-y||; x, y \in A\}$), and a point $x_0 \in A$ is called a nondiametral point of A if

$$r(x_0, A) = \sup\{||x_0 - y||; y \in A\} < diam(A).$$

A Banach space X is said to have normal structure if each bounded convex subset K of X with $diam(K) > 0$ contains a nondiametral point. This notion was introduced by Brodskii and Milman in [4], where it is shown that every weakly compact convex set which has this property contains a point which is fixed under surjective isometry.
A Banach space is said to have uniform normal structure (U.N.S.) if each bounded convex subset K of X with $diam(K) > 0$ contains a point x such that

$$r(x, K) = \sup\{||x-y||; y \in K\} \leq \alpha diam(K),$$

where α is independent of x and K. Let us recall that if a Banach space has U.N.S. then it is reflexive (see [2,8,12]). It has been generally known for many years that if $\epsilon_0(X) < 1$ then X has U.N.S. and an explicit proof of this fact is given in [6]. More on normal structure, can be found in [10,15].

Since our proof uses ultraproduct thechnique we start by making some basic definitions.
Let X be a Banach space and let $\mathcal{U}$ be a free ultrafilter over N (the set of natural integers). The ultraproduct space $\mathcal{X}$ of X is the quotient space of

$$l_\infty(X) = \{(x_n); x_n \in X \ \ and \ \ ||(x_n)|| = \sup_n ||x_n|| < \infty\}$$

$$by; \ \mathcal{N} = \{(x_n) \in l_\infty(X); \lim_{n\to\mathcal{U}} ||x_n|| = 0\}.$$

We shall not distinguish between $(x_n) \in l_\infty(X)$ and its class $(x_n) + \mathcal{N} \in \mathcal{X}$. Clearly

$$||(x_n)||_{\mathcal{X}} = \lim_{n\to\mathcal{U}} ||x_n||.$$

It is also clear that X is isometric to a subspace of $\mathcal{X}$ by the mapping $x \to (x, x, x, ...)$. Hence, we may assume that X is a subspace of $\mathcal{X}$. We will write $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ for the general elements of $\mathcal{X}$. When P is an hereditary property, we will say that X has super-P if and only if any ultraproduct $\mathcal{X}$ has P. For more details on ultraproduct thechnique we defer to [1,13]. Our first result compare super-normal structure and uniform normal structure.

Theorem 2. Let X be a Banach space. If X has super-normal structure then X has uniform normal structure.

Proof. Indeed assume that X fails to have U.N.S. Then there exists a sequence (K_n) of closed bounded convex subset of X with diameter equal to 1 such that $\lim\limits_{n\to\infty} r(K_n) = 1$, where

$$r(K) = \inf\{r(x, K); x \in K\}.$$

Let $\mathcal{X}$ be an ultraproduct of X and consider

$$\mathcal{K} = \{\mathbf{x} \in \mathcal{X}; \mathbf{x} = (x_n) \;\; with \;\; x_n \in K_n \;\; for\ every\ n \geq 1\}.$$

Let us show that for every $\mathbf{x} \in \mathcal{K}$ we have $r(\mathbf{x}, \mathcal{K}) = diam(\mathcal{K}) = 1$. Indeed let (ϵ_n) be a sequence of positive numbers such that $\lim\limits_{n\to\infty} \epsilon_n = 0$. Set $\mathbf{x} = (x_n)$. Then one can find (y_n) with $y_n \in K_n$ for every $n \geq 1$ such that $||x_n - y_n|| \geq r(K_n) - \epsilon_n$. Then, if $\mathcal{U}$ is the ultrafilter defining $\mathcal{X}$, we have

$$\lim_{n\to\mathcal{U}} ||x_n - y_n|| \geq \lim_{n\to\mathcal{U}} r(K_n) = 1.$$

Then if we put $\mathbf{y} = (y_n)$ we get $||\mathbf{x} - \mathbf{y}||_{\mathcal{X}} = 1 = diam(\mathcal{K})$. This clearly means that $\mathbf{x}$ is diametral point. Therefore $\mathcal{X}$ fails to have normal structure. The proof of Theorem 2 is therefore complete.

Remark. Recall that it is still unknown whether U.N.S. is equivalent to super-normal structure. Let us remark that if the Banach space is superreflexive then indeed these properties are equivalent. Then one can ask whether uniform normal structure implies superreflexivity. This problem is still open.

Let us now state our main result.

Theorem 3. Let X be a Banach space. Assume that its modulus of smoothness satisfies

$$(*) \quad \rho'_X(0) = \lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} < \frac{1}{2}.$$

Then X and X^* have super-normal structure.

Proof. Using Theorem 1, we know that the condition $(*)$ implies $\epsilon_0(X^*) < 1$ which therefore implies, by James'result that X^* is superreflexive. Hence X is also superreflexive. It is evidently true that for any ultraproduct $\mathcal{X}$ of X we have $\delta_X = \delta_{\mathcal{X}}$ and $\rho_X = \rho_{\mathcal{X}}$. Moreover it is known that X is superreflexive if and only if the dual of the ultraproduct space of X is the ultraproduct of the dual space of X (for more details see for example [13]). Therefore it is enough to show that X and X^* have normal structure.
Since $\epsilon_0(X^*) < 1$, then X^* has normal structure. Let us complete the proof by showing that X has normal structure. Assume to the contrary that there exists a diametral closed bounded convex subset not reduced to one point, say C. Then

$$r(x, C) = \sup\{||x - y||; y \in C\} = diam(C)$$

holds for every $x \in C$. Using Brodskii and Milman's characterization of normal structure property [4], we deduce that there exists a sequence $\{x_n\}$ in C, called diametral, such that

$$(**) \quad \lim_{n \to \infty} dist(x_{n+1}, conv\{x_i; i \leq n\}) = diam(C) = c.$$

Hence for every x in the closed convex hull of $\{x_n; n \geq 1\}$, we have

$$(***) \quad \lim_{n \to \infty} ||x_n - x|| = c.$$

Since any subsequence of $\{x_n\}$ satisfies also $(**)$ and $(***)$, so using the reflexivity of X we can assume that $\{x_n\}$ is weakly convergent, say to ω. Let $\tau \in (0, 1)$, then

$$\frac{1}{2}||x_n - x_m + \tau(\omega - x_m)|| + \frac{1}{2}||x_n - x_m - \tau(\omega - x_m)|| \leq ||x_n - x_m||(1 + \rho_X(\frac{\tau||\omega - x_m||}{||x_n - x_m||})).$$

Let n goes to infinity and using $(**)$ and $(***)$, we obtain

$$\frac{1}{2}\limsup_{n\to\infty}||x_n - x_m + \tau(\omega - x_m)|| \leq \frac{c}{2}(1 + 2\rho_X(\frac{\tau||\omega - x_m||}{c})),$$

and since $\{x_n\}$ converges weakly to ω, we get

$$\frac{1}{2}(1+\tau)||\omega - x_m|| \leq \frac{c}{2}(1 + 2\rho_X(\frac{\tau||\omega - x_m||}{c})).$$

Let m goes to infinity, we obtain

$$\frac{1}{2}(1+\tau)c \leq \frac{c}{2}(1 + 2\rho_X(\tau)).$$

Hence $\frac{\tau}{2} \leq \rho_X(\tau)$ since $c > 0$. This clearly implies that $\lim_{\tau\to 0}\frac{\rho_X(\tau)}{\tau} \geq \frac{1}{2}$ which contradicts our assumption. The proof is therefore complete.

As a consequence of Theorem 3, we deduce Baillon's result [3].

<u>**Theorem 4.**</u> Let X be a Banach space. Assume that its modulus of smoothness satisfyies

$$(*) \quad \rho'_X(0) = \lim_{\tau\to 0}\frac{\rho_X(\tau)}{\tau} < \frac{1}{2}.$$

Then X has the fixed point property, i.e. for any weakly compact convex subset K of X and any map $T : K \to K$ has a fixed point provided that T is nonexpansive (i.e. $||Tx - Ty|| \leq ||x - y||$ for every $x, y \in K$).

<u>Proof.</u> Since the condition on the modulus of smoothness implies normal structure, we use Kirk's result [9] to get the desired conclusion.

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