Fixed Point and Selection Theorems in Hyperconvex Spaces

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Abstract. It is shown that a set valued mapping T^* of a hyperconvex metric space M which takes values in the space of nonempty externally hyperconvex subsets of M always has a lipschitzian single valued selection T which satisfies $d(T(x), T(y)) \leq d_H(T^*(x), T^*(y))$ for all $x, y \in M$. (Here d_H denotes the usual Hausdorff distance.) This fact is used to show that the space of all bounded λ-lipschitzian self-mappings of M is itself hyperconvex. Several related results are also obtained.

1. Introduction

This paper focuses on external hyperconvexity, a concept which was introduced by Aronszajn and Panitchpakdi in their fundamental paper [**1**] on hyperconvexity. While hyperconvexity has be the subject of intense study, external hyperconvexity seems to have received relatively little attention.

From a functional analytic point of view interest in hyperconvex spaces stems from the fact that they include all L_∞ spaces. In fact it is known that a real Banach space is hyperconvex if and only if it is isometrically isomorphic to a space of continuous real-valued functions defined on a stonian space. See, e.g., [**5**] for details. Our main result, which extends the principal result of Sine [**9**], yields the fact that a lipschitzian set-valued mapping of a hyperconvex metric space into itself, taking externally hyperconvex values, always has a single valued selection which is lipshchitzian for the same constant. This is used to show that the family of all bounded λ-lipschitzian mappings of a hyperconvex space into itself is itself hyperconvex. Several related intersection theorems and fixed point theorems are also obtained.

We begin by describing the relevant notation and terminology. For a subset A of a metric space M we use $N_\varepsilon(A)$ to denote the closed ε-neighborhood of A. Thus

$$N_\varepsilon(A) = \{x \in M : dist(x, A) \leq \varepsilon\}.$$

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An *admissible* subset of M is a set of the form

$$\cap_i B(x_i; r_i)$$

where $\{B(x_i; r_i)\}$ is a family of closed balls centered at points $x_i \in M$ with respective radii r_i. The paper focuses on the following two fundamental concepts.

Definition 1. *A metric space M is said to be hyperconvex if given any family $\{x_\alpha\}$ of points of M and any family $\{r_\alpha\}$ of real numbers satisfying*

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

it is the case that $\cap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$.

Definition 2. *A subset E of a metric space M is said to be externally hyperconvex (relative to M) if given any family $\{x_\alpha\}$ of points in M and any family $\{r_\alpha\}$ of real numbers satisfying*

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta \text{ and } dist(x_\alpha, E) \leq r_\alpha$$

and it follows that $\cap_\alpha B(x_\alpha; r_\alpha) \cap E \neq \emptyset$.

The fundamental result of [**1**] asserts that a metric space M is hyperconvex if and only if it is *injective*. Thus M is hyperconvex if given any two metric spaces X and Y with Y a subspace of X, and any nonexpansive mapping $f : Y \to M$, then f has a nonexpansive extension $\tilde{f} : X \to M$. Basic results about injective metric spaces can be found in [**3**]. Also see [**11**].

Regarding externally hyperconvex spaces, it is shown in [**1**] that any admissible subset of a hyperconvex space M is externally hyperconvex relatively to M, and that the externally hyperconvex subsets of M are *proximinal* in M (thus if H is externally hyperconvex in M and if $x \in M$ then there exists $h \in H$ such that $d(x, h) = dist(x, H)$.) This fact is used below in the proof of Theorem 1. (It is known [**4**] that a hyperconvex subset of M need not be proximinal in M.)

In what follows we use $\mathcal{A}(M)$ to denote the family of all nonempty admissible subsets of M and $\mathcal{E}(M)$ to denote the family of all nonempty bounded subsets of M which are externally hyperconvex, in both instances endowed with the usual Hausdorff metric d_H. Recall that the distance between two closed subsets A, B of a metric space in the Hausdorff sense is given by

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}.$$

It is well known (and easy to see) that an admissible subset of a hyperconvex space is itself hyperconvex.

2. Main Results

The following selection theorem is the main result of this section. Using a different method, Sine [**9**] (Theorem 1) obtained this result in the special case $T^* : H \to \mathcal{A}(H)$ with T^* nonexpansive.

Theorem 1. *Let H be hyperconvex, and let $T^* : H \to \mathcal{E}(H)$. Then there exists a mapping $T : H \to H$ for which $T(x) \in T^*(x)$ for each $x \in H$ and for which $d(T(x), T(y)) \leq d_H(T^*(x), T^*(y))$ for each $x, y \in H$.*

Proof. Let $\mathfrak{F}$ denote the collection of all pairs (D, T), where $T : D \to H$, $T(d) \in T^*(d)\ \forall\, d \in D$, and $d(T(x), T(y)) \leq d_H(T^*(x), T^*(y))$ for each $x, y \in D$.

Notice that $\mathfrak{F} \neq \emptyset$ since $(\{x_0\}, T) \in \mathfrak{F}$ for any choice of $x_0 \in H$ and $T(x_0) \in T^*(x_0)$. Define an order relation on $\mathfrak{F}$ by setting

$$(D_1, T_1) \preceq (D_2, T_2) \Leftrightarrow D_1 \subset D_2 \text{ and } T_2\,|_{D_1} = T_1.$$

Let $\{(D_\alpha, T_\alpha)\}$ be an increasing chain in $(\mathfrak{F}, \preceq)$. Then it follows that $(\cup_\alpha D_\alpha, T) \in \mathfrak{F}$ where $T\,|_{D_\alpha} = T_\alpha$. By Zorn´s Lemma, $(\mathfrak{F}, \preceq)$ has a maximal element, say (D, T) . Assume $D \neq H$ and select $x_0 \in H \backslash D$. Set $\tilde{D} = D \cup \{x_0\}$ and consider the set

$$J = \cap_{x \in D} B(T(x); d_H(T^*(x), T^*(x_0))) \cap T^*(x_0).$$

Since $T^*(x_0) \in \mathcal{E}(H)$ for each $x \in H$, $J \neq \emptyset \Leftrightarrow$ for each $x \in D$

$$dist(T(x), T^*(x_0)) \leq d_H(T^*(x), T^*(x_0)).$$

Also, since $T^*(x_0)$ is a proximinal subset of H, the above is true $\Leftrightarrow$ for each $x \in D$,

$$B(T(x); d_H(T^*(x), T^*(x_0)) \cap T^*(x_0) \neq \emptyset.$$

By the definition of Hausdorff distance for each $\varepsilon > 0$

$$T^*(x) \subset N_{d_H(T^*(x), T^*(x_0)) + \varepsilon}(T^*(x_0)).$$

However by assumption $T(x) \in T^*(x)$ so it must be the case that for each $\varepsilon > 0$,

$$B(T(x); d_H(T^*(x), T^*(x_0)) + \varepsilon) \cap T^*(x_0) \neq \emptyset.$$

Since $T^*(x_0)$ is proximinal in H, this in turn implies

$$B(T(x); d_H(T^*(x), T^*(x_0))) \cap T^*(x_0) \neq \emptyset.$$

Thus we conclude $J \neq \emptyset$. Choose $y_0 \in J$ and define

$$\tilde{T}(x) = \begin{cases} y_0 & \text{if } x = x_0; \\ T(x) & \text{if } x \in D. \end{cases}$$

Since

$$d(\tilde{T}(x_0), \tilde{T}(x)) = d(y_0, T(x)) \leq d_H(T^*(x), T^*(x_0))$$

we conclude that $(D \cup \{x_0\}, \tilde{T}) \in \mathfrak{F}$ contradicting the maximality of (D, T). Therefore $D = H$. □

Corollary 1. *Let H be bounded and hyperconvex, and suppose $T^* : H \to \mathcal{E}(H)$ is nonexpansive. Then T^* has a fixed point, that is, there exists $x \in H$ such that $x \in T^*(x)$.*

Proof. If T^* is nonexpansive then the selection T assured by Theorem 1 is as well. The existence of a fixed point for T follows from the well known fixed point theorem of Sine [**7**] and Soardi [**10**]. □

The method of Theorem 1 also gives the following. In this theorem $Fix(T^*) = \{x \in H : x \in T^*(x)\}$.

Theorem 2. *Let H be hyperconvex, let $T^* : H \to \mathcal{E}(H)$ be nonexpansive and suppose $Fix(T^*) \neq \emptyset$. Then there exists a nonexpansive mapping $T : H \to H$ with $T(x) \in T^*(x)$ for each $x \in H$ and for which $Fix(T) = Fix(T^*)$.*

Proof. Let $\mathfrak{F}$ denote the collection of all pairs (D,T), where$D \supset Fix(T^*)$, $T : D \to H$, $T(d) \in T^*(d)$ for all $d \in D$, $T(x) = x$ for all $x \in Fix(T^*)$, and $d(T(x),T(y)) \leq d(x,y)$ for all $x,y \in D$. By assumption $(Fix(T^*), Id) \in \mathfrak{F}$, so $\mathfrak{F} \neq \emptyset$. The argument is now a simple modification of the proof of Theorem 1. Define an order relation on $\mathfrak{F}$ by setting

$$(D_1,T_1) \preceq (D_2,T_2) \Leftrightarrow D_1 \subset D_2 \text{ and } T_2\,|_{D_1} = T_1.$$

Let $\{(D_\alpha,T_\alpha)\}$ be an increasing chain in $(\mathfrak{F},\preceq)$. Then it follows that $(\cup_\alpha D_\alpha, T) \in \mathfrak{F}$ where $T\,|_{D_\alpha} = T_\alpha$. By Zorn´s Lemma, $(\mathfrak{F},\preceq)$ has a maximal element, say (D,T) . Assume $D \neq H$ and select $x_0 \in H \backslash D$. Set $\tilde{D} = D \cup \{x_0\}$ and consider the set

$$J = \cap_{x\in D} B(T(x); d(x,x_0)) \cap T^*(x_0).$$

Since $T^*(x_0) \in \mathcal{E}(H)$ for each $x \in H$, $J \neq \emptyset \Leftrightarrow$ for each $x \in D$

$$dist(T(x), T^*(x_0)) \leq d(x,x_0).$$

Also, since $T^*(x_0)$ is a proximinal subset of H, the above is true $\Leftrightarrow$ for each $x \in D$,

$$B(T(x); d(x,x_0)) \cap T^*(x_0) \neq \emptyset.$$

Using the definition of Hausdorff distance and the fact that T^* is nonexpansive, each $\varepsilon > 0$

$$T^*(x) \subset N_{d_H(T^*(x),T^*(x_0))+\varepsilon}(T^*(x_0)) \subset N_{d(x,x_0)+\varepsilon}(T^*(x_0)).$$

However by assumption $T(x) \in T^*(x)$ so it must be the case that for each $\varepsilon > 0$,

$$B(T(x); d(x,x_0)) + \varepsilon) \cap T^*(x_0) \neq \emptyset.$$

Since $T^*(x_0)$ is proximinal in H, this in turn implies

$$B(T(x); d(x,x_0)) \cap T^*(x_0) \neq \emptyset.$$

Thus we conclude $J \neq \emptyset$. Choose $y_0 \in J$ and define

$$\tilde{T}(x) = \begin{cases} y_0 & \text{if } x = x_0; \\ T(x) & \text{if } x \in D. \end{cases}$$

Since

$$d(\tilde{T}(x_0), \tilde{T}(x)) = d(y_0, T(x)) \leq d(x,x_0)$$

we conclude that $(D \cup \{x_0\}, \tilde{T}) \in \mathfrak{F}$ contradicting the maximality of (D,T). Therefore $D = H$. □

This in turn gives Corollary 3 of [**9**].

Corollary 2. *Let H be hyperconvex, let $T^* : H \to \mathcal{E}(H)$ is nonexpansive, and suppose $Fix(T^*) \neq \emptyset$. Then $Fix(T^*)$ is hyperconvex.*

Proof. The same conclusion holds for nonexpansive $T : H \to H$. □

In view of Corollary 2 $Fix(T^*)$ is a nonexpansive retract of H, and an approach of Lin and Sine [**6**] can be used to show that a retraction R of H onto $Fix(T^*)$ exists which commutes with the selection T of Theorem 2.

Theorem 1 also yields a set-valued 'Schauder' theorem.

Corollary 3. *Let H be compact and hyperconvex, and suppose $T^* : H \to \mathcal{E}(H)$ is continuous. Then T^* has a fixed point.*

As another application of Theorem 1 we show that the family of all bounded λ-lipschitzian functions of a hyperconvex space M into itself is itself hyperconvex. For two such functions we define distance in the usual way, that is, if $f, g : M \to M$, set

$$d(f, g) = \sup_{x \in M} d(f(x), g(x)).$$

For this result we also need the following lemma due to R. Sine.

Lemma 1. *If M is hyperconvex and if $D = \cap_\alpha B(z_\alpha; r_\alpha)$, then for any $\rho > 0$*

$$N_\rho(D) = \cap_\alpha B(z_\alpha; r_\alpha + \rho).$$

Theorem 3. *Let M be hyperconvex and for $\lambda > 0$ let $\mathfrak{F}_\lambda$ denote the family of all bounded λ-lipschitzian functions of M into M. Then $\mathfrak{F}_\lambda$ is itself a hyperconvex space.*

Proof. Suppose $\{f_\alpha\} \subset \mathfrak{F}_\lambda$ and $\{r_\alpha\} \subset \mathbb{R}$ satisfy $d(f_\alpha, f_\beta) \leq r_\alpha + r_\beta$. Then for each $x \in M$

$$d(f_\alpha(x), f_\alpha(x)) \leq r_\alpha + r_\beta$$

so in view of the hyperconvexity of M

$$J(x) := \cap_\alpha B(f_\alpha(x); r_\alpha) \neq \emptyset.$$

We show that $d_H(J(x), J(y)) \leq \lambda d(x, y)$ for each $x, y \in M$. To see this it clearly suffices to show that for each $x, y \in M$

$$J(x) \subset N_{\lambda d(x,y)}(J(y)).$$

However if $z \in J(x)$ then for each α

$$\begin{aligned} d(z, f_\alpha(y)) &\leq d(z, f_\alpha(x)) + d(f_\alpha(x), f_\alpha(y)) \\ &\leq d(z, f_\alpha(x)) + \lambda d(x, y) \\ &\leq r_\alpha + \lambda d(x, y). \end{aligned}$$

Using Sine's Lemma we now have

$$z \in \cap_\alpha B(f_\alpha(y); r_\alpha + \lambda d(x, y)) = N_{\lambda d(x,y)}(J(y))$$

In view of Theorem 1 it is possible to select $f(x) \in J(x)$ for each $x \in M$ so that $f \in \mathfrak{F}_\lambda$. Since $f \in \cap_\alpha B(f_\alpha; r_\alpha)$, $\mathfrak{F}_\lambda$ is hyperconvex. □

This leads to the following.

Corollary 4. *Let M be a bounded hyperconvex metric space and let $f \in \mathcal{F}_1$. Then the family*

$$R := \{r \in \mathcal{F}_1 : r(M) \subset Fix(f)\}$$

is a nonexpansive retract of $\mathcal{F}_1$.

Proof. The mapping $T_f : \mathcal{F}_1 \to \mathcal{F}_1$ defined via the formula $T_f(g) = f \circ g$ is nonexpansive and has a nonempty fixed point set which is a nonexpansive retract of $\mathcal{F}_1$ ([**2**]). But $r \in fix(T_f) \Leftrightarrow r \in R$. □

We conclude this section with the following observation.

Proposition 1. *Let M be a metric space and suppose A is an externally hyperconvex subset of a hyperconvex space M. Then $N_\varepsilon(A)$ is externally hyperconvex (in M) for each $\varepsilon > 0$.*

Proof. Let $\{x_\alpha\} \subset M$ and $\{r_\alpha\} \subset \mathbb{R}$ satisfy $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $dist(x_\alpha, N_\varepsilon(A)) \leq r_\alpha$. The latter inequality implies $dist(x_\alpha, A) \leq r_\alpha + \varepsilon$. Since A is externally hyperconvex this in turn implies

$$A \cap (\cap_\alpha B(x_\alpha; r_\alpha + \varepsilon)) \neq \emptyset.$$

By Sine's Lemma

$$\cap_\alpha B(x_\alpha; r_\alpha + \varepsilon) = N_\varepsilon(\cap_\alpha B(x_\alpha; r_\alpha));$$

thus

$$A \cap N_\varepsilon(\cap_\alpha B(x_\alpha; r_\alpha)) \neq \emptyset.$$

Therefore

$$N_\varepsilon(A) \cap (\cap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$$

and we conclude that $N_\varepsilon(A)$ is externally hyperconvex in M. □

3. Hyperconvex intersections

While the intersection of two admissible subsets of a given hyperconvex space is again admissible, in general it is not the case that the intersection of two hyperconvex subspaces of a hyperconvex space is itself hyperconvex, even if one of them is admissible. However the following is true.

Lemma 2. *Let H be a hyperconvex metric space. Suppose $E \subset H$ is externally hyperconvex relative to H and suppose A is an admissible subset of H. Then $E \cap A$ is externally hyperconvex relative to H.*

Proof. Suppose $\{x_\alpha\}$ and $\{r_\alpha\}$ satisfy $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $dist(x_\alpha, E \cap A) \leq r_\alpha$. Since A is admissible, $A = \cap_{i \in I} B(x_i; r_i)$ and since $dist(x_\alpha, E \cap A) \neq \emptyset$ it follows that $d(x_\alpha, x_i) \leq r_\alpha + r_i$ for each $i \in I$. Also, since $A \subset B(x_i; r_i)$, it follows that $dist(x_i, E \cap A) \leq r_i$ and that $d(x_i, x_j) \leq r_i + r_j$ for each $i, j \in I$. Therefore by external hyperconvexity of E

$$(\cap_i B(x_i; r_i))(\cap_\alpha B(x_\alpha, r_\alpha)) \cap E = \cap_\alpha B(x_\alpha, r_\alpha) \cap (A \cap E) \neq \emptyset.$$

□

This leads to the following.

Theorem 4. *Let $\{H_i\}$ be a descending chain of nonempty externally hyperconvex subsets of a bounded hyperconvex space H. Then $\cap_i H_i$ is nonempty and externally hyperconvex in H.*

Proof. A result of Baillon [**2**] assures that $D := \cap_i H_i \neq \emptyset$. To see that D is externally hyperconvex let $\{x_\alpha\} \subset H$ and $\{r_\alpha\} \subset \mathbb{R}$ satisfy $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ and $dist(x_\alpha, D) \leq r_\alpha$. Since H is hyperconvex we know that $A := \cap_\alpha B(x_\alpha; r_\alpha) \neq \emptyset$. Also, since $dist(x_\alpha, D) \leq r_\alpha$ we have $dist(x_\alpha, H_i) \leq r_\alpha$ for each i, so by external hyperconvexity of H_i we conclude $A \cap H_i \neq \emptyset$ for each i. By Lemma 2 $\{A \cap H_i\}$ is descending chain of nonempty hyperconvex subsets of H, so again by [**2**] $\cap_i(A \cap H_i) = A \cap D \neq \emptyset$. □

Another consequence of Lemma 2 provides yet further evidence of the ubiquitous nature of hyperconvexity.

THEOREM 5. *Let H be a hyperconvex metric space and suppose $T^*: H \to \mathcal{E}(H)$. Then the family $S(T^*)$, consisting of all mappings $T: H \to H$ for which $T(x) \in T^*(x)$ and $d(T(x), T(y)) \le d_H(T^*(x), T^*(y))$ for each $x, y \in H$, is hyperconvex.*

PROOF. Suppose $\{T_\alpha\} \subset S(T^*)$ and $\{r_\alpha\}$ satisfy $d(T_\alpha, T_\beta) \le r_\alpha + r_\beta$. Then since $T^*(x)$ is hyperconvex

$$J(x) := (\cap_\alpha B(T_\alpha(x); r_\alpha)) \cap T^*(x) \neq \emptyset$$

for each $x \in H$. Moreover, by Lemma 2, $J(x) \in \mathcal{E}(H)$. Therefore by Theorem 1 the mapping $x \longmapsto J(x)$ has a selection T which satisfies $d(T(x), T(y) \le d_H(T^*(x), T^*(y))$ for each $x, y \in H$. Thus $T \in \cap_\alpha B(T_\alpha; r_\alpha) \cap S(T^*)$. □

It is easy to see that the intersection of two hyperconvex subspaces of a hyperconvex space is always hyperconvex if one of the subspaces has *unique metric segments*. Using this fact it is possible to prove the following.

THEOREM 6. *Suppose M is a hyperconvex metric space and suppose H is a bounded hyperconvex subspace of M which has unique metric segments. Then any nonexpansive mapping $T: H \to M$ which satisfies*

$$\inf\{d(x, T(x)) : x \in H\} = 0$$

always has a fixed point.

PROOF. Since M is hyperconvex T has a nonexpansive extension $\tilde{T}: M \to M$. Let

$$F_n := \{x \in M : d(x, T(x)) \le 1/n\},\ n = 1, 2, \cdots.$$

It is known [**8**] that each of the sets F_n is a hyperconvex subspace of M, and clearly $H_n := F_n \cap H \neq \emptyset$ for each n. Since H has unique metric segments H_n is hyperconvex, so $\cap_{n=1}^\infty H_n \neq \emptyset$ ([**2**]). Clearly each point of $\cap_{n=1}^\infty H_n$ is a fixed point of T. □

We conclude this section with a simple observation about admissible sets which is another easy consequence of Sine's Lemma.

PROPOSITION 2. *Suppose M is hyperconvex, and let $U = \cap_{i \in I} B(x_i; r_i)$ and $V = \cap_{i \in I} B(y_i; r_i)$. Then*

$$d_H(U, V) \le \sup\{d(x_i, y_i) : i \in I\}.$$

PROOF. Let $\rho := \sup\{d(x_i, y_i) : i \in I\}$ and let $x \in U$. Then

$$d(x, y_i) \le d(x, x_i) + d(x_i, y_i) \le r_i + \rho.$$

Thus

$$x \in \cap_i B(y_i; r_i + \rho) = N_\rho(\cap_i B(y_i; r_i) = N_\rho(V)$$

and we conclude $U \subset N_\rho(V)$. Reversing the roles of U and V gives the conclusion. □

4. Examples

It might be interesting to note that Theorem 6 fails without the assumption of unique metric segments, even if H is an admissible set.

EXAMPLE 1. *Let B denote the unit ball in ℓ_∞ and let $H = B(\mathbf{z}_1; 1) \cap B(\mathbf{z}_2; 1)$ where*

$$\mathbf{z}_1 = (1, 0, 0, \cdots) \text{ and } \mathbf{z}_2 = (-1, 0, 0, \cdots).$$

Let $\mathbf{x} = (0, x_2, x_3, \cdots) \in H$ and define $T : H \to B$ by the formula

$$T(\mathbf{x}) = (1 - \sup_{2 \le i < \infty} (1 - \frac{1}{i}) \, |x_i| \, , (1 - \frac{1}{2}) x_2, \cdots, (1 - \frac{1}{n}) x_n, \cdots).$$

Then T is both nonexpansive and fixed point free. However, if $\mathbf{e}^n$ denotes the standard unit vector basis, then

$$\|\mathbf{e}^n - T(\mathbf{e}^n)\| = \frac{1}{n}, \quad n = 2, 3, \cdots.$$

Several facts about externally hyperconvex subsets can easily be deduced from simple examples in $\mathbb{R}^2_\infty$. As we mentioned earlier, it is shown in [**1**] that every admissible subset of a hyperconvex space is externally hyperconvex. It is not difficult to show that an externally hyperconvex subset of $\mathbb{R}^2_\infty$ is necessarily an admissible subset of $\mathbb{R}^2_\infty$. However it is easy to see that an externally hyperconvex subset of a hyperconvex space need not always be admissible.

EXAMPLE 2. *Let H be the solid rectangle in $\mathbb{R}^2_\infty$ with corners $(\pm 2, \pm 1)$, and let E be the line interval $[-2, 2]$. Then E is externally hyperconvex relative to H but E is not an admissible subset of H (although clearly E is an admissible subset of $\mathbb{R}^2_\infty$). Notice that H itself is hyperconvex because it is an admissible subset of the hyperconvex space $\mathbb{R}^2_\infty$.*

The following example exhibits another curious property of external hyperconvexity.

EXAMPLE 3. *Let $H = \{(x, x) : 0 \le x \le 1\}$. Then H is not externally hyperconvex relative to $\mathbb{R}^2_\infty$. On the other hand, H is externally hyperconvex relative to $H \cup \{(1, 0)\}$.*

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