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On asymptotic pointwise contractions in modular function spaces

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ABSTRACT

We discuss the existence of fixed points of asymptotic pointwise contractions in modular function spaces.

1. Introduction

The purpose of this paper is to give an outline of the fixed point theory for asymptotic pointwise contractions defined on some subsets of modular function spaces which are natural generalizations of both function and sequence variants of many important, from application perspective, spaces like Lebesgue, Kothe, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz, Calderon–Lozanovskii spaces and many others. The importance for applications consists in the richness of the structure of modular function spaces, that – besides being Banach spaces (or F-spaces in more general settings) – are equipped with modular equivalents of norm or metric notions. They are also equipped with almost everywhere convergence and convergence in submeasure. In many cases, particularly in applications to integral operators, approximation and fixed point theory, modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts. There are also important results that can be proved only using the apparatus of modular function spaces. From this perspective, the fixed point theory in modular function spaces should be considered as complementary to the fixed point theory in normed and metric spaces.

The theory of contractions and nonexpansive mappings defined on convex subsets of Banach spaces has been well developed since the 1960s [1–4], and generalized to other metric spaces [5–7], and modular function spaces [8,9]. The corresponding fixed point results were then extended to larger classes of mappings like asymptotic mappings [10,11], pointwise contractions [12] and asymptotic pointwise contractions and nonexpansive mappings [13–15].

The most common approach in the Banach space fixed point theory is to assume a weak compactness of the set on which a nonlinear mapping is defined, or alternatively to assume reflexivity of the Banach space, which in itself – via the Milman Theorem – guarantees the weak compactness of the bounded sets. Questions may be asked whether the theory of modular function spaces provides similarly, general methods for the consideration of fixed point properties. We believe that this

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paper demonstrates the existence of such a general theory. In our view, the most important, from the fixed point theory viewpoint, characteristics of reflexive spaces sits in the following purely geometric characterization of reflexive spaces: every nonincreasing sequence of nonempty, convex, bounded sets has a nonempty intersection. In the context of modular function spaces we follow [9] and call this property (R). In the theory developed in this paper, the property (R) plays a central role as a modular equivalent of the Banach space reflexivity. The property (R) also aligns well to the metric equivalents of reflexivity defined by the notions of compact convexity structures, [15].

Another thread of thought present in this paper is to use a modular version of the Opial property which together with a compactness in the sense of convergence almost everywhere gives us existence of a fixed point for pointwise contractions. It is worthwhile mentioning that the modular version of the Opial property is satisfied in a large class of modular function spaces (including all Orlicz and Musielak–Orlicz spaces) while the Banach space, weak convergence version of Opial condition is failed even in cases like L^p spaces for $1 < p < \infty$, $p \neq 2$, [16,17].

The paper is organized as follows:

(a) Section 2 provides necessary preliminary material and establishes the terminology and key notation conventions.
(b) Section 3 presents fixed point theorems for pointwise contractions in modular function spaces. Theorem 3.1 assumes uniform continuity of ρ. A more general assumption on ρ of the Strong Opial property gives a fixed point result for pointwise contractions defined in sets that are compact in the sense of the convergence almost everywhere (Theorem 3.2).
(c) Section 4 deals with asymptotic pointwise contractions in modular function spaces. Again, we prove first a fixed point theorem for uniform continuous ρ (Theorem 4.1). Theorem 4.2 provides the fixed point result for asymptotic pointwise contractions defined in sets that are compact in the sense of the convergence almost everywhere under the Strong Opial property assumption on ρ.

2. Preliminaries

Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A.

Definition 2.1. Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a convex and even function. We say that ρ is a regular convex function pseudomodular if:

(i) $\rho(0) = 0$;
(ii) ρ is monotone, i.e. $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(iii) ρ is orthogonally subadditive, i.e. $\rho(f 1_{A\cup B}) \leq \rho(f 1_A) + \rho(f 1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}$;
(iv) ρ has the Fatou property, i.e. $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(v) ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty \rho\text{-a.e}\}, \tag{2.1}$$

where each $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

Definition 2.2. Let ρ be a regular function pseudomodular.

(1) We say that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0 \rho$-a.e.;
(2) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0 \rho$-a.e.

The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\Re$.

Let us denote $\rho(f, E) = \rho(f 1_E)$ for $f \in \mathcal{M}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudomodular in the sense of Def. 2.1.1 in [18] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function spaces as per the framework defined by Kozlowski in [18–20], see also Musielak [21].

Remark 2.1. We limit ourselves to convex function modulars in this paper. However, omitting convexity in Definition 2.1 or replacing it by s-convexity would lead to the definition of nonconvex or s-convex regular function pseudomodulars, semimodulars and modulars as in [18].

Definition 2.3 ([19,20,18]). Let ρ be a convex function modular.

(a) A modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

(b) The following formula defines a norm in L_ρ (frequently called Luxemurg norm):

$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \le 1\}.$$

In the following theorem we recall some of the properties of modular function spaces that will be used later on in this paper.

Theorem 2.1 ([19,20,18]). *Let $\rho \in \Re$.*

(1) *L_ρ, $\|f\|_\rho$ is complete and the norm $\|\cdot\|_\rho$ is monotone w.r.t. the natural order in $\mathcal{M}$.*
(2) *$\|f_n\|_\rho \to 0$ if and only if $\rho(\alpha f_n) \to 0$ for every $\alpha > 0$.*
(3) *If $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$ then there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $g_n \to 0\rho$-a.e.*
(4) *If $\{f_n\}$ converges uniformly to f on a set $E \in \mathcal{P}$ then $\rho(\alpha(f_n - f), E) \to 0$ for every $\alpha > 0$.*
(5) *Let $f_n \to f\,\rho$-a.e. There exists a nondecreasing sequence of sets $H_k \in \mathcal{P}$ such that $H_k \uparrow \Omega$ and $\{f_n\}$ converges uniformly to f on every H_k (Egoroff Theorem).*
(6) *$\rho(f) \le \liminf \rho(f_n)$ whenever $f_n \to f\,\rho$-a.e. (Note: this property is equivalent to the Fatou Property.)*
(7) *Defining $L^0_\rho = \{f \in L_\rho; \rho(f, \cdot) \text{ is order continuous}\}$ and $E_\rho = \{f \in L_\rho;\ \lambda f \in L^0_\rho \text{ for every } \lambda > 0\}$ we have:*
    (a) *$L_\rho \supset L^0_\rho \supset E_\rho$,*
    (b) *E_ρ has the Lebesgue property, i.e. $\rho(\alpha f, D_k) \to 0$ for $\alpha > 0, f \in E_\rho$ and $D_k \downarrow \emptyset$.*
    (c) *E_ρ is the closure of $\mathcal{E}$ (in the sense of $\|\cdot\|_\rho$).*

The following definition plays an important role in the theory of modular function spaces.

Definition 2.4. Let $\rho \in \Re$. We say that ρ has the Δ_2-property if $\sup_n \rho(2f_n, D_k) \to 0$ whenever $D_k \downarrow \emptyset$ and $\sup_n \rho(f_n, D_k) \to 0$.

Theorem 2.2. *Let $\rho \in \Re$. The following conditions are equivalent:*

(a) *ρ has Δ_2,*
(b) *L^0_ρ is a linear subspace of L_ρ,*
(c) *$L_\rho = L^0_\rho = E_\rho$,*
(d) *if $\rho(f_n) \to 0$ then $\rho(2f_n) \to 0$,*
(e) *if $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$ then $\|f_n\|_\rho \to 0$, i.e. the modular convergence is equivalent to the norm convergence.*

We will also use another type of convergence which is situated between norm and modular convergence. It is defined, among other important terms, in the following definition.

Definition 2.5. Let $\rho \in \Re$.

(a) We say that $\{f_n\}$ is ρ-convergent to f and write $f_n \to 0\ (\rho)$ if and only if $\rho(f_n - f) \to 0$.
(b) A sequence $\{f_n\}$ where $f_n \in L_\rho$ is called ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.
(c) A set $C \subset L_\rho$ is called ρ-closed if for any sequence $\{f_n\}$ in C, the convergence $f_n \to f\ (\rho)$ implies that f belongs to C.
(d) A set $C \subset L_\rho$ is called ρ-bounded if $\sup\{\rho(f - g); f \in C, g \in C\} < \infty$.
(e) A set $C \subset L_\rho$ is called ρ-a.e. closed if for any $\{f_n\}$ in C which ρ-a.e. converges to some f, then we must have $f \in C$;
(f) A set $C \subset L_\rho$ is called ρ-a.e. compact if for any $\{f_n\}$ in C, there exists a subsequence $\{f_{n_k}\}$ which ρ-a.e. converges to some $f \in C$.
(g) Let $f \in L_\rho$ and $C \subset L_\rho$. The ρ-distance between f and C is defined as

$$d_\rho(f, C) = \inf\{\rho(f - g); g \in C\}.$$

Let us note that ρ-convergence does not necessarily imply ρ-Cauchy condition. Also, $f_n \to f$ does not imply in general $\lambda f_n \to \lambda f,\ \lambda > 1$. Using Theorem 2.1 it is not difficult to prove the following

Proposition 2.1. *Let $\rho \in \Re$.*

(i) *L_ρ is ρ-complete,*
(ii) *ρ-balls $B_\rho(x, r) = \{y \in L_\rho; \rho(x - y) \le r\}$ are ρ-closed and ρ-a.e. closed.*

Let us give the modular definitions of pointwise contractions, asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings. The definitions are straightforward generalizations of their norm and metric equivalents, [10,13–15].

Definition 2.6. Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called a pointwise contraction if there exists $\alpha : C \to [0, 1)$ such that

$$\rho(T(f) - T(g)) \le \alpha(f)\rho(f - g) \quad \text{for any } f, g \in C,\ n \ge 1.$$

Definition 2.7. Let $\rho \in \Re$ and let $C \subset L_\rho$ be nonempty and ρ-closed. A mapping $T : C \to C$ is called an asymptotic pointwise mapping if there exists a sequence of mappings $\alpha_n : C \to [0, 1]$ such that

$$\rho(T^n(f) - T^n(g)) \le \alpha_n(f)\rho(f - g) \quad \text{for any } f, g \in C.$$

(i) If $\{\alpha_n\}$ converges pointwise to $\alpha : C \to [0, 1)$, then T is called asymptotic pointwise contraction.
(ii) If $\limsup_{n\to\infty} \alpha_n(f) \le 1$ for any $f \in C$, then T is called asymptotic pointwise nonexpansive.
(iii) If $\limsup_{n\to\infty} \alpha_n(f) \le k$ for any $f \in C$ with $0 < k < 1$, then T is called strongly asymptotic pointwise contraction.

The following result will be useful throughout this work.

Theorem 2.3. *Let us assume that $\rho \in \Re$. Let $K \subset L_\rho$ be nonempty, ρ-closed and ρ-bounded. Let $T : K \to K$ be a pointwise ρ-contraction or asymptotic pointwise ρ-contraction. Then T has at most one fixed point $x_0 \in K$. Moreover if x_0 is a fixed point of T, then the orbit $\{T^n(x)\}$ converges to x_0 for any $x \in K$.*

Proof. Since pointwise ρ-contraction implies asymptotic pointwise ρ-contraction, then we only focus on asymptotic pointwise ρ-contraction mappings. Indeed let T be an asymptotic pointwise ρ-contraction and $u, v \in C$ are fixed point of T. Then we have

$$\rho(T^n(u), T^n(v)) = \rho(u, v) \le \alpha_n(u)\rho(u, v),$$

for any $n \ge 1$. If we let $n \to \infty$, we will get

$$\rho(u, v) \le \alpha(u)\rho(u, v).$$

Since $\alpha(u) < 1$, we conclude that $\rho(u, v) = 0$ or $u = v$. This proves the first part of Theorem 2.3. Next let x_0 be a fixed point of T and $x \in C$. Let us prove that $\{T^n(x)\}$ converges to x_0. Indeed we have

$$\rho(T^{n+m}(x), T^n(x_0)) = \rho(T^{n+m}(x), x_0) \le \alpha_n(x_0)\rho(T^m(x), x_0),$$

for any $n, m \ge 1$. Hence

$$\limsup_{m\to\infty} \rho(T^{n+m}(x), x_0) \le \limsup_{m\to\infty} \alpha_n(x_0)\rho(T^m(x), x_0).$$

Since $\limsup_{m\to\infty} \rho(T^{n+m}(x), x_0) = \limsup_{m\to\infty} \rho(T^m(x), x_0)$, we get

$$\limsup_{m\to\infty} \rho(T^m(x), x_0) \le \alpha_n(x_0) \limsup_{m\to\infty} \rho(T^m(x), x_0),$$

for any $n \ge 1$. If we let $n \to \infty$, we obtain

$$\limsup_{m\to\infty} \rho(T^m(x), x_0) \le \alpha(x_0) \limsup_{m\to\infty} \rho(T^m(x), x_0).$$

Since $\alpha(x_0) < 1$, we get

$$\limsup_{m\to\infty} \rho(T^m(x), x_0) = 0.$$

Clearly we can derive the same equality where lim sup is replaced by lim inf which implies the desired conclusion:

$$\lim_{m\to\infty} \rho(T^m(x), x_0) = 0. \quad \square$$

3. Pointwise contractions in modular function spaces

To begin with let us recall the following standard definitions.

Definition 3.1. We will say that the function modular ρ is uniformly continuous if for every $\varepsilon > 0$ and $L > 0$ there exists $\delta > 0$ such that

$$|\rho(g) - \rho(h + g)| \le \varepsilon \quad \text{if } \rho(h) \le \delta \quad \text{and} \quad \rho(g) \le L. \tag{3.1}$$

Definition 3.2. A function $\lambda : C \to [0, \infty]$, where $C \subset L_\rho$ is nonempty and ρ-closed, is called ρ-lower semicontinuous if for any $\alpha > 0$, the set $C_\alpha = \{f \in C; \lambda(f) \le \alpha\}$ is ρ-closed.

It can be proved that ρ-lower semicontinuity is equivalent to the condition

$$\lambda(f) \le \liminf_{n\to\infty} \lambda(f_n) \quad \text{provided } f, f_n \in C, \text{ and } \rho(f - f_n) \to 0.$$

The following lemma shows how uniform continuity of the modular ρ is related to the ρ-lower semicontinuity of the associated limit of a sequence of ρ-radii being modular analogue of the Chebyshev radii in metric spaces.

Definition 3.3. Let $x \in L_\rho$ and let $C \subset L_\rho$ be nonempty. Then by the ρ-Chebyshev radius of C with respect to $x \in L_\rho$ we understand

$$r_\rho(x, C) = \sup\{\rho(x - y); y \in C\}.$$

Lemma 3.1. *Let $\rho \in \Re$ be uniformly continuous. Let $K \subset L_\rho$ be nonempty, convex, ρ-closed and ρ-bounded. Let $\{K_n\}$ be a nonincreasing sequence of nonempty, convex, ρ-closed subsets of K with a nonempty intersection, denoted by K_∞. Then the function r defined as $r(x) = \inf_{n\ge 0} r_\rho(x, K_n)$ is ρ-lower semicontinuous in K_∞.*

Proof. Let $\alpha > 0$, denote $C_\alpha = \{x \in K_\infty; r(x) \le \alpha\}$. We need to prove that C_α is ρ-closed. Let a sequence $\{x_k\} \subset C_\alpha$ and $\rho(x_0 - x_k) \to 0$. We need to prove that $x_0 \in C_\alpha$.

To every $n \ge 1$ there exists $y_n \in K_n$ such that

$$r_\rho(x_0, K_n) - \frac{1}{n} \le \rho(x_0 - y_n) \le r_\rho(x_0, K_n).$$

Since $r_\rho(x_0, K_n) \to r(x_0)$ as $n \to \infty$, it follows that

$$r(x_0) = \lim_{n\to\infty} \rho(x_0 - y_n). \tag{3.2}$$

By ρ-boundedness of K, there exists $L = \sup_n \rho(x_0 - y_n) < \infty$. Let us fix an arbitrary $\varepsilon > 0$. By uniform continuity of ρ there exists $\delta > 0$ such that

$$|\rho(g) - \rho(h + g)| \le \varepsilon \quad \text{if } \rho(h) \le \delta \text{ and } \rho(g) \le L. \tag{3.3}$$

Since $\rho(x_0 - x_k) \to 0$, there exists $p \ge 1$ such that $\rho(x_0 - x_k) \le \delta$ for $k \ge p$. Using (3.3) with $h = x_0 - x_p$ and $g = y_n - x_0$ we have

$$|\rho(y_n - x_0) - \rho(y_n - x_p)| \le \varepsilon \quad \text{for every } n \ge 1. \tag{3.4}$$

Since

$$\rho(y_n - x_p) \le \sup\{\rho(y - x_p); y \in K_n\} = r_\rho(x_p, K_n),$$

we get

$$\rho(y_n - x_0) \le r_\rho(x_p, K_n) + \varepsilon,$$

for any $n \ge 1$. If we let $n \to \infty$, we obtain

$$r(x_0) \le r(x_p) + \varepsilon \le \alpha + \varepsilon.$$

Since ε was arbitrary, we deduce finally that $r(x_0) \le \alpha$, i.e. $x_0 \in C_\alpha$ as claimed. □

Remark 3.1. Let us mention that uniform continuity holds for a large class of function modulars. For instance, it can be proved that in Orlicz spaces over a finite atomless measure [22] or in sequence Orlicz spaces [23] the uniform continuity of the Orlicz modular is equivalent to the Δ_2-type condition.

The following property plays in the theory of modular function spaces a role similar to the reflexivity in Banach spaces (see e.g. [9]).

Definition 3.4. We say that L_ρ has property (R) if and only if every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.

In [9] examples of modular function spaces which posses property (R) are given.

The following result plays an important role in the proof of the Fixed Point Theorem 3.1.

Lemma 3.2. *Let us assume that $\rho \in \Re$ has property (R). Let $K \subset L_\rho$ be nonempty, convex, ρ-closed and ρ-bounded. If $\varphi : K \to [0, \infty)$ is a ρ-lower semicontinuous convex function, then there exists $x_0 \in K$ such that*

$$\varphi(x_0) = \inf\{\varphi(x); x \in K\}.$$

Proof. Let $m = \inf\{\varphi(x); x \in K\}$. The assumptions on φ imply $m < \infty$. For any $n \geq 1$, set

$$K_n = \left\{x \in K; \varphi(x) \leq m + \frac{1}{n}\right\}.$$

Clearly K_n is not empty and is a convex set because φ is a convex function. Also, K_n is ρ-closed since φ is ρ-lower semicontinuous. Since ρ satisfies property (R), then

$$K_\infty = \bigcap_{n \geq 1} K_n \neq \emptyset.$$

For $x_0 \in K_\infty$ there holds

$$\varphi(x_0) = \inf\{\varphi(x); x \in K\},$$

as claimed. □

We are ready now to prove our fixed point theorem which assumes the uniform continuity of the function modular ρ.

Theorem 3.1. *Let us assume that $\rho \in \Re$ is uniformly continuous and has property (R). Let $K \subset L_\rho$ be nonempty, convex, ρ-closed and ρ-bounded. Let $T : K \to K$ be a pointwise ρ-contraction. Then T has a unique fixed point $x_0 \in K$. Moreover the orbit $\{T^n(x)\}$ converges to x_0 for any $x \in K$.*

Proof. Using Theorem 2.3, it is enough to show that T has a fixed point. For any subset A of L_ρ, denote by $\overline{\text{conv}}_\rho(A)$ the intersection of all ρ closed convex subsets of L_ρ which contains A. Set $K_0 = K$, and

$$K_{n+1} = \overline{\text{conv}}_\rho(T(K_n)), \quad \text{for any } n \geq 0.$$

Since K is T-invariant, then we have $K_{n+1} \subset K_n$, for any $n \geq 0$. Since ρ satisfies the property (R), then

$$K_\infty = \bigcap K_n \neq \emptyset. \tag{3.5}$$

Clearly we have $T(K_\infty) \subset K_\infty$. We claim that K_∞ is reduced to one point. Indeed for any $x \in K_\infty$, note that

$$r_\rho(x, K_{n+1}) \leq r_\rho(x, K_n),$$

for any $n \geq 0$, where $r_\rho(x, C) = \sup\{\rho(x - y); y \in C\}$. Set

$$r(x) = \inf_{n \geq 0} r_\rho(x, K_n) = \lim_{n \to \infty} r_\rho(x, K_n).$$

Since $K_n \subset B_\rho(x, r_\rho(x, K_n))$, we get $K_\infty \subset B_\rho(x, r_\rho(x, K_n))$, for any $n \geq 0$. Hence

$$r_\rho(x, K_\infty) \leq r(x). \tag{3.6}$$

By Lemma 3.1 we know that $r(x)$ is ρ-lower semicontinuous. Lemma 3.2 implies then that the infimum of $r(x)$ on K_∞ is attained at a point $x_0 \in K_\infty$, i.e. $r(x_0) = \inf\{r(x); x \in K_\infty\}$.

Let us prove now that

$$r(T(x_0)) \leq \alpha(x_0) r(x_0). \tag{3.7}$$

Indeed we have

$$K_n \subset B_\rho(x_0, r_\rho(x_0, K_n)),$$

for any $n \geq 0$. Since T is a ρ-pointwise contraction, it follows then that

$$T(K_n) \subset B_\rho(T(x_0), \alpha(x_0) r_\rho(x_0, K_n)).$$

Since ρ is convex and ρ-balls are ρ-closed, it follows that

$$\overline{\text{conv}}_\rho(T(K_n)) \subset B_\rho(T(x_0), \alpha(x_0) r_\rho(x_0, K_n)),$$

which implies

$$K_{n+1} \subset B_\rho(T(x_0), \alpha(x_0) r_\rho(x_0, K_n)).$$

Hence $r_\rho(T(x_0), K_{n+1}) \leq \alpha(x_0)\, r_\rho(x_0, K_n)$, which implies

$$r(T(x_0)) \leq \alpha(x_0) r(x_0) \tag{3.8}$$

which proves (3.7).

Since $\alpha(x_0) < 1$ and $T(x_0) \in X_\infty$, we get $r(x_0) = \inf\{r(x); x \in K_\infty\}$ must be equal to zero. In view of (3.6), this forces the following

$$r_\rho(x_0, K_\infty) = 0.$$

Because modulars are equal zero only at zero, we get $K_\infty = \{x_0\}$. Since K_∞ is T-invariant, x_0 is a fixed point of T. □

Using ρ-a.e. Strong Opial property of the function modular we can prove now the next fixed point theorem which does not assume uniform continuity of ρ.

Definition 3.5. We will say that L_ρ satisfies the ρ-a.e. Strong Opial property (or shortly SO-Property) if for every $\{f_n\} \in L_\rho$ which is ρ-a.e. convergent to 0 such that there exists a $\beta > 1$ for which

$$\sup\{\rho(\beta f_n)\} < \infty, \tag{3.9}$$

the following equality holds for any $g \in E_\rho$

$$\liminf_{n\to\infty} \rho(f_n + g) = \liminf_{n\to\infty} \rho(f_n) + \rho(g). \tag{3.10}$$

Remark 3.2. Note that the ρ-a.e. Strong Opial property implies ρ-a.e. Opial property (see the paper by Khamsi [24] for definition of the Opial property in modular function spaces).

Remark 3.3. Also, note that, in virtue of Theorem 2.1 in [24], every convex, orthogonally additive function modular ρ has the ρ-a.e. Strong Opial property. Let us recall that ρ is called orthogonally additive if $\rho(f, A \cup B) = \rho(f, A) + \rho(f, B)$ whenever $A \cap B = \emptyset$.

Remark 3.4. The ρ-a.e. Strong Opial property can be also defined and proved for nonconvex regular function modulars, e.g. for some s-convex modulars ($s < 1$) like L^s for $0 < s < 1$, [24,17].

We start with the following lemma which explains the role the ρ-a.e. Strong Opial property plays in proving existence of minimal elements for some real functions being a pointwise limit or lim sup of a sequence of functions defined in subsets of L_ρ. Typically, we would use this technique to types or limits of Chebyshev radii. This minimization argument will be essential in the proof of the fixed point theorems using the ρ-a.e. Strong Opial property.

Lemma 3.3. *Let $\rho \in \Re$. Assume that L_ρ has the* ρ-a.e. *Strong Opial property. Let $K \subset E_\rho$ be a nonempty,* ρ-a.e. *compact subset such that there exists $\beta > 1$ such that $\delta_\rho(\beta K) = \sup\{\rho(\beta(x - y)); x, y \in K\} < \infty$. Let $C \subset K$ be a nonempty* ρ-a.e. *closed subset. For any $n \geq 1$, let $\lambda_n : C \to [0, \infty)$ be such that for any $y \in C$, there exists a sequence $\{y_n\} \subset K$ such that, for every $n \geq 1$, there holds*

$$\lambda_n(y) - \frac{1}{n} \leq \rho(y - y_n),$$

and $\rho(x - y_n) \leq \lambda_n(x)$, for every $x \in C$ and every $n \geq 1$. Let $\lambda(x) = \limsup_{n\to\infty} \lambda_n(x)$, for any $x \in C$. Then there exists $x_0 \in C$ at which λ attains infimum, i.e.

$$\lambda(x_0) = \inf\{\lambda(x); x \in C\}.$$

Proof. First note that $\inf\{\lambda(x); x \in C\} \leq \delta_\rho(K) \leq \delta_\rho(\beta K) < \infty$. Hence there exists $\{x_n\} \subset C$ such that

$$\lambda_0 = \lim_{n\to\infty} \lambda(x_n) = \inf\{\lambda(x); x \in C\}.$$

Without loss of any generality we may assume $\{x_n\}$ ρ-a.e. converges to some $x_0 \in C$ since K is ρ-a.e. compact and C is ρ-a.e. closed. By the hypothesis, for any $n \geq 1$, there exists $y_n \in K$ such that

$$\rho(x_0 - y_n) \geq \lambda_n(x_0) - \frac{1}{n}. \tag{3.11}$$

Without loss of generality we can assume that

$$\lambda(x_0) = \lim_{n\to\infty} \lambda_n(x_0). \tag{3.12}$$

There exists a subsequence $\{y_{\varphi(n)}\}$ of $\{y_n\}$ which ρ-a.e. converges to some $y_0 \in K$. By the ρ-a.e. Strong Opial property we get

$$\liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_m) = \liminf_{n\to\infty} \rho(y_{\varphi(n)} - y_0) + \rho(y_0 - x_m), \tag{3.13}$$

for any $m \geq 0$. Since

$$\liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_m) \leq \limsup_{n\to\infty} \rho(y_n - x_m) \leq \limsup_{n\to\infty} \lambda_n(x_m) = \lambda(x_m) \tag{3.14}$$

we conclude via (3.13) that

$$\liminf_{m\to\infty} \lambda(x_m) \geq \liminf_{n\to\infty} \rho(y_{\varphi(n)} - y_0) + \liminf_{m\to\infty} \rho(y_0 - x_m). \tag{3.15}$$

Using the ρ-a.e. Strong Opial property again, to $\{x_m - x_0\}$ this time, we get

$$\liminf_{m\to\infty} \rho(y_0 - x_m) = \liminf_{m\to\infty} \rho(x_m - x_0) + \rho(x_0 - y_0) \tag{3.16}$$

which implies

$$\liminf_{m\to\infty} \lambda(x_m) \geq \liminf_{n\to\infty} \rho(y_{\varphi(n)} - y_0) + \liminf_{m\to\infty} \rho(x_m - x_0) + \rho(x_0 - y_0). \tag{3.17}$$

Hence

$$\liminf_{m\to\infty} \lambda(x_m) \geq \liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_0) + \liminf_{m\to\infty} \rho(x_m - x_0), \tag{3.18}$$

which implies

$$\liminf_{m\to\infty} \lambda(x_m) \geq \liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_0). \tag{3.19}$$

Using (3.11) we get

$$\liminf_{n\to\infty} \rho(y_{\varphi(n)} - x_0) \geq \liminf_{n\to\infty} \lambda_{\varphi(n)}(x_0) = \lambda(x_0),$$

where the last equality follows from (3.12). This combined with (3.19) yields $\lambda_0 \geq \lambda(x_0)$. By the definition of λ_0 we have then

$$\lambda_0 = \lambda(x_0) = \inf\{\lambda(x); x \in C\} \tag{3.20}$$

as claimed. □

Theorem 3.2. *Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $K \subset E_\rho$ be a nonempty, ρ-a.e. compact convex subset such that there exists $\beta > 1$ such that $\delta_\rho(\beta K) = \sup\{\rho(\beta(x - y)); x, y \in K\} < \infty$. Then any $T : K \to K$ pointwise ρ-contraction has a unique fixed point $x_0 \in K$. Moreover the orbit $\{T^n(x)\}$ converges to x_0, for any $x \in K$.*

Proof. Using Theorem 2.3, it is enough to show that T has a fixed point. For any subset A of L_ρ, denote by $\overline{\text{conv}}_{\rho\text{-a.e.}}(A)$ the intersection of all ρ-a.e. closed convex subsets of L_ρ which contains A. The assumptions on K imply that K is ρ-bounded. Set $K_0 = K$, and

$$K_{n+1} = \overline{\text{conv}}_{\rho\text{-a.e.}}(T(K_n)), \quad \text{for any } n \geq 0.$$

Since K is T invariant, then we have $K_{n+1} \subset K_n$, for any $n \geq 0$. Since K is ρ-a.e. compact, then

$$K_\infty = \bigcap K_n \neq \emptyset. \tag{3.21}$$

Clearly we have $T(K_\infty) \subset K_\infty$. We claim that K_∞ is reduced to one point. Indeed for any $x \in K_\infty$, note that

$$r_\rho(x, K_{n+1}) \leq r_\rho(x, K_n),$$

for any $n \geq 0$, where $r_\rho(x, C) = \sup\{\rho(x - y); y \in C\}$, because the balls are ρ-a.e. closed by the Fatou property. Set

$$r(x) = \inf_{n\geq 0} r_\rho(x, K_n) = \lim_{n\to\infty} r_\rho(x, K_n).$$

Since $K_n \subset B_\rho(x, r_\rho(x, K_n))$, we get $K_\infty \subset B_\rho(x, r_\rho(x, K_n))$, for any $n \geq 0$. Hence

$$r_\rho(x, K_\infty) \leq r(x). \tag{3.22}$$

The existence of $x_0 \in K_\infty$ such that $r(x_0) = \inf\{r(x); x \in K_\infty\}$ follows from Lemma 3.3 applied with $\lambda(x) = r(x)$, $C = K_\infty$, $\lambda_n(x) = r_\rho(x, K_n)$. Note for a given $y \in K_\infty$ there exists $y_n \in K_n$ such that

$$\lambda_n(y) - \frac{1}{n} = r_\rho(y, K_n) - \frac{1}{n} \leq \rho(y - y_n).$$

Therefore the assumptions of Lemma 3.3 are satisfied. Let us prove now that

$$r(T(x_0)) \leq \alpha(x_0) r(x_0). \tag{3.23}$$

Indeed we have

$$K_n \subset B_\rho(x_0, r_\rho(x_0, K_n)),$$

for any $n \geq 0$. Since T is ρ-pointwise contraction, it follows then that

$$T(K_n) \subset B_\rho(T(x_0), \alpha(x_0) r_\rho(x_0, K_n)).$$

Since ρ is convex and has the Fatou property, it follows that

$$\overline{\text{conv}}_{\rho\text{-a.e.}}(T(K_n)) \subset B_\rho(T(x_0), \alpha(x_0) r_\rho(x_0, K_n)),$$

which implies

$$K_{n+1} \subset B_\rho(T(x_0), \alpha(x_0) r_\rho(x_0, K_n)).$$

Hence $r_\rho(T(x_0), K_{n+1}) \leq \alpha(x_0)\, r_\rho(x_0, K_n)$, which implies

$$r(T(x_0)) \leq \alpha(x_0) r(x_0) \tag{3.24}$$

which proves (3.23).

Since $\alpha(x_0) < 1$ and $T(x_0) \in X_\infty$, we get $r(x_0) = \inf\{r(x); x \in K_\infty\}$ must be equal to zero. In view of (3.22), this forces the following

$$r_\rho(x_0, K_\infty) = 0.$$

Because modulars are equal zero only at zero, we get $K_\infty = \{x_0\}$. Since K_∞ is T-invariant, x_0 is a fixed point of T. □

Please note that combining Theorem 3.2 with Remark 3.3 we have the next result.

Theorem 3.3. *Let L_ρ be a modular function space where $\rho \in \Re$ is orthogonally additive. Let $K \subset E_\rho$ be a nonempty, ρ-*a.e. *compact convex subset such that $\delta_\rho(\beta K) = \sup\{\rho(\beta(x-y)); x, y \in K\} < \infty$, for some $\beta > 1$. Then any $T : K \to K$ pointwise ρ-contraction has a unique fixed point $x_0 \in K$. Moreover the orbit $\{T^n(x)\}$ converges to x_0, for any $x \in M$.*

4. Asymptotic pointwise contractions in modular function spaces

We begin this section by introducing a notion of a ρ-type.

Definition 4.1. Let $K \subset L_\rho$ be convex and ρ-bounded.

(1) A function $\tau : K \to [0, \infty]$ is called a (ρ)-type if there exists a sequence $\{y_m\}$ of elements of K such that for any $z \in K$ there holds

$$\tau(z) = \limsup_{m\to\infty} \rho(y_m - z).$$

(2) A sequence $\{g_n\}$ is called a minimizing sequence of τ if

$$\lim_{n\to\infty} \tau(g_n) = \inf\{\tau(f); f \in C\}.$$

Note that τ is convex provided ρ is convex. We will use types for proving an existence of fixed points for asymptotic pointwise ρ-contractions. We will start with the following easy but important result.

Lemma 4.1. *Let $\rho \in \Re$ be uniformly continuous. Let $K \subset L_\rho$ be nonempty, convex, ρ-closed and ρ-bounded. Then any ρ-type $\tau : K \to [0, \infty]$ is ρ-lower semicontinuous in K.*

Proof. Let τ be a ρ-type. Let $\alpha > 0$, denote $C_\alpha = \{x \in K; \tau(x) \leq \alpha\}$. We need to prove that C_α is ρ-closed. Without loss of generality we can assume that C_α is nonempty. Let a sequence $\{x_k\} \subset C_\alpha$ be such that $\rho(x_0 - x_k) \to 0$ with $x_0 \in K$. We need to prove that $x_0 \in C_\alpha$, i.e. that $\tau(x_0) \leq \alpha$. Let $\{y_m\}$ be a sequence that defines τ. By ρ-boundedness of K, there exists $L = \sup_m \rho(x_0 - y_m) < \infty$. Let us fix an arbitrary $\varepsilon > 0$. By uniform continuity of ρ there exists $\delta > 0$ such that

$$|\rho(g) - \rho(h+g)| \leq \varepsilon \quad \text{if } \rho(h) \leq \delta \text{ and } \rho(g) \leq L. \tag{4.1}$$

Since $\rho(x_0 - x_k) \to 0$, there exists $p \geq 1$ such that $\rho(x_0 - x_k) \leq \delta$ for $k \geq p$. Using (4.1) with $h = x_0 - x_p$ and $g = y_n - x_0$ we have

$$|\rho(y_n - x_0) - \rho(y_n - x_p)| \leq \varepsilon \quad \text{for every } n \geq 1. \tag{4.2}$$

By the definition of τ, we have then that $\tau(x_0) \leq \alpha + 2\varepsilon$ because $x_p \in C_\alpha$. Since ε was chosen arbitrarily we have finally $\tau(x_0) \leq \alpha$ as claimed. □

We will use Lemma 4.1 in the proof of the following fixed point theorem.

Theorem 4.1. *Let us assume that $\rho \in \Re$ is uniformly continuous and has property (R). Let $K \subset L_\rho$ be nonempty, convex, ρ-closed and ρ-bounded. Let $T : K \to K$ be an asymptotic pointwise ρ-contraction. Then T has a unique fixed point $x_0 \in K$. Moreover the orbit $\{T^n(x)\}$ converges to x_0 for any $x \in K$.*

Proof. Using Theorem 2.3, it is enough to show that T has a fixed point. Indeed let us fix $x \in K$ and define the ρ-type by

$$\tau(u) = \limsup_{n\to\infty} \rho(T^n(x) - u), \tag{4.3}$$

for $u \in K$. By Lemma 4.1 the ρ-type τ is ρ-lower semicontinuous in K. By Lemma 3.2 then there exists $x_0 \in K$ such that

$$\tau(x_0) = \inf\{\tau(x); x \in K\}. \tag{4.4}$$

Let us prove that $\tau(x_0) = 0$. Indeed, for any $n, m \geq 1$ we have

$$\rho(T^{n+m}(x) - T^m(x_0)) \leq \alpha_m(x_0)\rho(T^n(x) - x_0). \tag{4.5}$$

If let n go to infinity, we get

$$\tau(T^m(x_0)) \leq \alpha_m(x_0)\tau(x_0), \tag{4.6}$$

which implies

$$\tau(x_0) = \inf\{\tau(x); x \in K\} \leq \tau(T^m(x_0)) \leq \alpha_m(x_0)\tau(x_0). \tag{4.7}$$

Passing with m to infinity we get $\tau(x_0) \leq \alpha(x_0)\tau(x_0)$ which forces $\tau(x_0) = 0$ as $\alpha(x_0) < 1$. Hence, $\rho(T^n(x) - x_0) \to 0$ as $n \to \infty$. By the ρ-continuity of T, this forces x_0 to be a fixed point of T. □

Similarly as we did in the case of pointwise contractions, we will use ρ-a.e. Strong Opial property of the function modular to prove the next fixed point theorem for asymptotic pointwise contractions.

Theorem 4.2. *Let $\rho \in \Re$. Assume that L_ρ has the ρ-a.e. Strong Opial property. Let $K \subset E_\rho$ be a nonempty, ρ-a.e. compact convex subset such that $\delta_\rho(\beta K) = \sup\{\rho(\beta(x-y)); x, y \in K\} < \infty$, for some $\beta > 1$. Then any $T : K \to K$ asymptotic pointwise ρ-contraction has a unique fixed point $x_0 \in K$. Moreover the orbit $\{T^n(x)\}$ converges to x_0, for any $x \in K$.*

Proof. Using Theorem 2.3, it is enough to show that T has a fixed point. Indeed let us fix $x \in K$ and define the ρ-type by

$$\tau(u) = \limsup_{n\to\infty} \rho(T^n(x) - u), \tag{4.8}$$

for $u \in K$. By Lemma 3.3 applied with $\lambda(u) = \tau(u)$, $C = K$, $\lambda_n(u) = \rho(T^n(x) - u)$, and with $y_n = T^n(x)$ chosen for all $y \in K$, there exists $x_0 \in K$ such that

$$\tau(x_0) = \inf\{\tau(x); x \in K\}. \tag{4.9}$$

Using the same argument as in the proof of Theorem 4.1, we will get $\tau(x_0) = 0$. Hence, $\rho(T^n(x) - x_0) \to 0$ as $n \to \infty$. By the ρ-continuity of T, this forces x_0 to be a fixed point of T. □

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