Problem 40. Let F_n be a nest of connected compact sets ($F_{n+1} \subset F_n$) in a metric space (M,d). Show that $K = \cap F_n$ is connected. Discuss the necessity of compactness.

Answer. Clearly if $K = \emptyset$, the conclusion is true. Therefore we will assume that $K \neq \emptyset$. Since K is a closed subset of the compact set F_1, then K is also compact. Assume that K is not connected. Then there exist two open sets U and V such that:

1. $K \subset U \cup V$,
2. $K \cap U \neq \emptyset$ and $K \cap V \neq \emptyset$,
3. $K \cap U \cap V = \emptyset$.

Set $K_1 = K \cap U^c$, where U^c is the complement of U, and $K_2 = K \cap V^c$. It is easy to see that K_1 and K_2 are nonempty closed subsets of K such that

$$K = K_1 \cup K_2 \ \text{ and } \ K_1 \cap K_2 = \emptyset.$$

Therefore K_1 and K_2 are disjoint compact subsets of K. Next we show that there exists $\varepsilon_0 > 0$ such that for any $x \in K_1$ and $y \in K_2$, we have $d(x,y) \geq \varepsilon_0$. Indeed assume not, then for any $\varepsilon > 0$, there exist $x \in K_1$ and $y \in K_2$ such that $d(x,y) < \varepsilon$. In particular, there exist (x_n) in K_1 and (y_n) in K_2 such that $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$. Since K_1 is compact, then we can assume without loss of any generality that (x_n) is convergent to some $x \in K_1$. It is clear that (y_n) also converges to x. Since K_2 is closed, then we have $x \in K_2$. This is the sought contradiction with $K_1 \cap K_2 = \emptyset$. This proves the existence of ε_0. Next set

$$O_1 = \bigcup_{x \in K_1} D(x, \varepsilon_0/3) \ \text{ and } \ O_2 = \bigcup_{x \in K_2} D(x, \varepsilon_0/3),$$

where $D(x,r) = \{y \in M; d(x,y) < r\}$ is the open disc or ball in M. As union of open sets, both O_1 and O_2 are nonempty open subsets of M. Note that:

1. $K_1 \subset O_1$, and $K_2 \subset O_2$,
2. $O_1 \cap O_2 = \emptyset$.

Next we prove that there exists $n_0 \geq 1$ such that $F_{n_0} \subset O_1 \cup O_2$. Indeed assume not, then for any $n \geq 1$ we have $F_n \not\subset O_1 \cup O_2$. This will imply

$$F_n \cap O_1^c \cap O_2^c \neq \emptyset$$

for any $n \geq 1$. Since the sequence $\{F_n\}$ is nested and both O_1 and O_2 are open, the compactness assumption will force

$$\Big(O_1 \cup O_2\Big)^c \cap K = O_1^c \cap O_2^c \cap K \neq \emptyset.$$

This will contradict the assumption $K = K_1 \cup K_2 \subset O_1 \cup O_2$. Therefore there exists $n_0 \geq 1$ such that $F_{n_0} \subset O_1 \cup O_2$. Since $K_1 \subset F_{n_0} \cap O_1$ and $K_2 \subset F_{n_0} \cap O_2$, we get

$$F_{n_0} \cap O_1 \neq \emptyset, \ \text{ and } F_{n_0} \cap O_2 \neq \emptyset.$$

The disjointness property of O_1 and O_2 gives the desired contradiction of the connectedness of F_{n_0}.

<u>**Remark.**</u> The Nested property is crucial since the intersection of two compact connected sets may not be connected. The example may be easily found in $\mathbb{R}^2$, not in $\mathbb{R}$ where the result is true. The above conclusion is false when we only assume that the sequence $\{F_n\}$ are closed. Indeed, take

$$F_n = \mathbb{R}^2 \setminus \left\{(x, y); |x| < \frac{1}{n} \text{ and } |y| < n\right\}.$$

It is easy to see that F_n are closed and

$$\bigcap_{n\geq 1} F_n = \mathbb{R}^2 \setminus \left\{(0, y); y \in \mathbb{R}\right\}.$$

Moreover each F_n is connected while their intersection is not.